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Recurrence Relations for Moment Generating Functions of Generalized Order Statistics from Some Specific Continuous Distributions

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ABSTRACT

In this paper, we establish some recurrence relations satisfied by the moment generating functions of generalized order statistics from pth order exponential distribution, left truncated logistic distribution and extreme value distribution. In all the above distributions, the corresponding recurrence relations for single and product moments of generalized order statistics have also been deduced.

Key words and Phrases: Generalized order statistics, recurrence relations, pth order exponential distribution, left truncated logistic distribution, extreme value distribution.

1 Introduction

Generalized order statistics (gos) have been introduced and extensively studied in Kamps (1995a, b) as a unified theoretical set-up which contains a variety of models of ordered random variables with different interpretations. Examples of such

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models are: Ordinary order statistics, Sequential order statistics, Progressive type-II censored order statistics, Record values, kth record values and Pfeifer's records. There is no natural interpretation of generalized order statistics in terms of observed random samples but these models can be effectively applied in life testing and reliability analysis, medical and life time data, and models related to software reliability analysis, etc. The common approach makes it possible to define several distributional properties at once. The structural similarities of these models are based on the similarity of their joint density function.

Let $\{X_n, n \geq 1\}$be a sequence of absolutely continuous, independent and identically distributed random variables with cdf $F(x) = P(X \leq x)$ and pdf $f(x)$. Assume $k > 0$, $n \in \{2, 3, \ldots\}$, $\widetilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in R^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$, such that $\gamma_r = k + n - r + M_r > 0$ for all $r \in \{1, 2, \ldots, n-1\}$. Then $X(r, n, \widetilde{m}, k)$, $r = 1, 2, \ldots, n$, are called gos if their joint pdf is given by

$$f^{X(1,n,\widetilde{m},k),X(2,n,\widetilde{m},k),\ldots,X(n,n,\widetilde{m},k)}(x_1, x_2, \ldots, x_n)$$
$$= k \left(\prod_{j=1}^{n-1} \gamma_j \right) \left(\prod_{i=1}^{n-1} (1 - F(x_i))^{m_i} f(x_i) \right) (1 - F(x_n))^{k-1} f(x_n), \qquad (1.1)$$

where $F^{-1}(0+) < x_1 \leq x_2 \leq \ldots \leq x_r < F^{-1}(1)$.

By choosing appropriate values of parameters, we get the distribution of a few very common statistics as shown in Table 1. The joint pdf of first r, *gos* is given by:

$$f^{X(1,n,\widetilde{m},k),X(2,n,\widetilde{m},k),\ldots,X(r,n,\widetilde{m},k)}(x_1, x_2, \ldots, x_r)$$
$$= c_{r-1} \left(\prod_{i=1}^{r-1} (1 - F(x_i))^{m_i} f(x_i) \right) (1 - F(x_r))^{k+n-r+M_r-1} f(x_r), \qquad (1.2)$$

where $c_{r-1} = \prod_{j=1}^{r} \gamma_j$ and $F^{-1}(0+) < x_1 \leq x_2 \leq \ldots \leq x_n < F^{-1}(1)$.

We now consider two cases:

Case I: $m_1 = m_2 = \ldots = m_{n-1} = m$.

Case II: $\gamma_i \neq \gamma_j$, $i \neq j$, $i, j = 1, 2, \ldots, n-1$.

Table 1:

S.No.	Choice of parameters for $i = 1, 2, \ldots, n$	Generalized order statistics become
1	$\gamma_i = n - i + 1$, $m_1 = m_2 = \ldots = m_{n-1} = 0$ and $k = 1$	Ordinary order statistics
2	$\gamma_i = k$, $m_1 = m_2 = \ldots = m_{n-1} = -1$, $k \in N$	kth record values
3	$\gamma_i = (n - i + 1)\alpha_i$, $\alpha_i > 0$	Sequential order statistics
4	$\gamma_i = \alpha - i + 1$, $\alpha > 0$	Order statistics with non-integer sample size
5	$\gamma_i = \beta_i$, $\beta_i > 0$	Pfeifer's record values
6	$m_i \in N_o$, $k \in N$	Progressively type-II right censored order statistics

For Case I, the rth *gos* will be denoted by $X(r, n, m, k)$. The pdf of $X(r, n, m, k)$ is given by

$$f^{X(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!}(1 - F(x))^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)), \quad x \in R \tag{1.3}$$

and the joint pdf of $X(r, n, m, k)$ and $X(s, n, m, k)$, $1 \le r < s \le n$, is given by

$$\begin{aligned} f^{X(r,n,m,k),X(s,n,m,k)}(x, y) = {} & \frac{c_{s-1}}{(r-1)!(s-r-1)!}((1 - F(x))^m f(x)) g_m^{r-1}(F(x)) \\ & \times [h_m(F(y)) - h_m(F(x))]^{s-r-1}(1 - F(y))^{\gamma_s - 1} f(y), \\ & -\infty < x < y < \infty, \end{aligned} \tag{1.4}$$

where

$$c_{r-1} = \prod_{j=1}^{r} \gamma_j, \quad \gamma_j = k + (n - j)(m + 1), \quad r = 1, 2, \ldots, n,$$

$$g_m(x) = h_m(x) - h_m(0), x \in (0, 1)$$

and

$$h_m(x) = \begin{cases} -\dfrac{(1-x)^{m+1}}{m+1}, & m \neq -1, \\ -\log(1 - x), & m = -1. \end{cases} \tag{1.5}$$

(see, Kamps, 1955a, b).

For Case II, the rth *gos* will be denoted by $X(r,n,\widetilde{m},k)$. The pdf of $X(r,n,\widetilde{m},k)$ is given by

$$f^{X(r,n,\widetilde{m},k)}(x) = c_{r-1}\ f(x)\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i-1}, \quad x \in R. \tag{1.6}$$

Also, the joint pdf of $X(r,n,\widetilde{m},k)$ and $X(s,n,\widetilde{m},k)$, $1 \le r < s \le n$, is given by

$$f^{X(r,n,\widetilde{m},k),X(s,n,\widetilde{m},k)}(x,y)$$
$$= c_{s-1}\left\{\sum_{i=r+1}^{s} a_i^r(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}\right\}\left\{\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i}\right\}\frac{f(x)}{1-F(x)}\frac{f(y)}{1-F(y)},$$
$$-\infty < x < y < \infty, \tag{1.7}$$

where $c_{s-1} = \prod_{j=1}^{s}\gamma_j$, $\gamma_j = k+n-j+M_j$, $s = 1,2,\ldots,n$.

Further, it can be proved that

(i) $a_i(r) = \prod_{j(\neq i)=1}^{r}(\gamma_j-\gamma_i)^{-1}, \quad 1 \le i \le r \le n$

(ii) $a_i^r(s) = \prod_{j(\neq i)=r+1}^{s}(\gamma_j-\gamma_i)^{-1}, \quad r+1 \le i \le s \le n$

(iii) $a_i(r) = (\gamma_{r+1}-\gamma_i)\ a_i(r+1)$

(iv) $c_r = c_{r-1}\gamma_{r+1}$

(v) $\sum_{i=1}^{r+1} a_i(r+1) = 0$

(vi) $$\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i} = \frac{(1-F(x))^{\gamma_r}}{(r-1)!}g_m^{r-1}(F(x)) \tag{1.8}$$

(vii) $$\sum_{i=r+1}^{s} a_i^r(s)\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_i}$$
$$= \frac{(1-F(x))^{-(m+1)(s-r-1)}}{(s-r-1)!}\left(\frac{1-F(y)}{1-F(x)}\right)^{\gamma_s}(h_m(F(y))-h_m(F(x)))^{s-r-1} \tag{1.9}$$

In this paper we shall make use of a few identities, stated in the form of Lemmas 2.1, 2.3, 3.1 and 3.2, respectively, in Athar and Islam (2004), for Borel measurable functions $\omega(x)$ and $\omega(x,y)$ with support (α,β), which are quoted below:

$$\text{(i)} \quad E(\omega(X(r,n,m,k))) - E(\omega(X(r-1,n,m,k))) = \frac{c_{r-2}}{(r-1)!}\int_{\alpha}^{\beta}\frac{\partial\omega(x)}{\partial x}(\bar{F}(x))^{\gamma_r} g_m^{r-1}(F(x))dx \tag{1.10}$$

$$\text{(ii)} \quad E(\omega(X(r,n,\widetilde{m},k))) - E(\omega(X(r-1,n,\widetilde{m},k))) = c_{r-2}\int_{\alpha}^{\beta}\frac{\partial\omega(x)}{\partial x}\left[\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i}\right]dx \tag{1.11}$$

$$\text{(iii)} \quad E(\omega(X(r,n,m,k),X(s,n,m,k))) - E(\omega(X(r,n,m,k),X(s-1,n,m,k))) = \frac{c_{s-2}}{(r-1)!(s-r-1)!}\int_{\alpha}^{\beta}\int_{x}^{\beta}\frac{\partial\omega(x,y)}{\partial y}(\bar{F}(x))^{m} f(x) g_m^{r-1}(F(x)) \times [h_m(F(y))-h_m(F(x))]^{s-r-1}(\bar{F}(y))^{\gamma_s}dydx \tag{1.12}$$

$$\text{(iv)} \quad E(\omega(X(r,n,\widetilde{m},k),X(s,n,\widetilde{m},k))) - E(\omega(X(r,n,\widetilde{m},k),X(s-1,n,\widetilde{m},k))) = c_{s-2}\int_{\alpha}^{\beta}\int_{x}^{\beta}\frac{\partial\omega(x,y)}{\partial y}\left[\sum_{i=1}^{r} a_i(r)(\bar{F}(x))^{\gamma_i}\right]\frac{f(x)}{\bar{F}(x)}\left[\sum_{i=r+1}^{s} a_i^r(s)\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i}\right]dydx, \tag{1.13}$$

where $\bar{F}(x) = 1 - F(x)$.

The moments of order statistics have generated considerable interest in the recent years. Several recurrence relations and identities satisfied by the single as well as product moments of order statistics have been obtained by various authors in the past. These relations help in reducing the quantum of computations involved. Joshi (1978, 1982) established recurrence relations for exponential distribution with unit mean and were further extended by Balakrishnan and Joshi (1984) for doubly truncated exponential distribution. For linear-exponential distribution, Balakrishnan and Malik (1986) derived similar type of relations which were extended to doubly truncated linear-exponential distribution by Mohie El-Din et al. (1997) and Saran and Pushkarna (1999). Nain (2010 a,b) obtained recurrence relations for ordinary order statistics and kth record values from pth order exponential and generalized

Weibull distributions, respectively. Saran and Nain (2012 b) derived recurrence relations for moments of kth record values from doubly truncated pth order exponential and generalized Weibull distributions.

The recurrence relations for the moments of generalized order statistics based on non-identically distributed random variables were developed by Kamps (1995 a,b). Pawlas and Szynal (2001) obtained recurrence relations for single and product moments of generalized order statistics from Pareto, generalized Pareto and Burr distributions. Saran and Pandey (2004, 2009) obtained recurrence relations for single and product moments of generalized order statistics from linear-exponential and Burr distributions. Saran and Pandey (2011) obtained recurrence relations for marginal and joint moment generating functions of dual (lower) generalized order statistics from inverse Weibull distribution. Saran and Nain (2012 a) obtained recurrence relations for single and product moments of generalized order statistics from doubly truncated pth order exponential distribution. Nain (2014) obtained recurrence relations for single and product moments of generalized order statistics from extreme value distribution.

In this paper, we establish some recurrence relations satisfied by the marginal and joint moment generating functions of generalized order statistics from pth order exponential distribution, left truncated logistic distribution and extreme value distribution. These distributions have increasing failure rate for large values of x and have many applications in reliability analysis. In Sections 2 we introduce pth order exponential distribution (POED) and establish recurrence relations among their moment generating functions. Section 3 is devoted to left truncated logistic distribution (LTLD) and recurrence relations among their moment generating functions. In Section 4, we define the extreme value distribution (EVD) and state the corresponding recurrence relations among their moment generating functions. For all the above distributions, the corresponding recurrence relations for single and product moments of generalized order statistics have also been deduced as special cases. The results so obtained are generalized versions of some of the recurrence relations obtained by Kumar (2010), and Saran and Pandey (2004, 2009).

Table 2:

S.No.	Choice of parameters	Name of the pth order exponential distribution
1	$\alpha_0 > 0$ and $\alpha_j = 0,\ j \geq 1$	Exponential
2	$\alpha_0 = 0,\ \alpha_1 > 0$ and $\alpha_j = 0,\ j \geq 2$	Rayleigh
3	$\alpha_p > 0$ and $\alpha_j = 0,\ 0 \leq j \leq p-1$	Weibull
4	$\alpha_0 \neq 0,\ \alpha_1 > 0$ and $\alpha_j = 0,\ j \geq 2$	Linear-exponential
5	$\alpha_j = 1,\ j \geq 0$ and $p = \infty$	Power series
6	$\alpha_0 = 1,\ \alpha_j = \frac{(-1)^j}{j!},\ j \geq 1$ and $p = \infty$	Left truncated logistic
7	$\alpha_j = \frac{\alpha^{j+1}}{j!},\ j \geq 0$ and $p = \infty$	Standard extreme value

2 Recurrence relations for MGF's of generalized order statistics from POED

A random variable X is said to have the pth order exponential distribution if its probability density function is of the form

$$f(x) = \left(\sum_{j=0}^{p} \alpha_j x^j\right) e^{-\sum\limits_{j=0}^{p} \alpha_j \frac{x^{j+1}}{j+1}}, \quad 0 \leq x < \infty, \tag{2.1}$$

where $\alpha_p > 0$ for some fixed positive integer p, and the cumulative distribution function is of the form

$$F(x) = 1 - e^{-\left(\alpha_0 x + \alpha_1 \frac{x^2}{2} + \alpha_2 \frac{x^3}{3} + \ldots + \alpha_p \frac{x^{p+1}}{p+1}\right)}. \tag{2.2}$$

The characterizing differential equation for pth order exponential distribution is given by

$$f(x) = \left(\sum_{j=0}^{p} \alpha_j x^j\right)(1 - F(x)). \tag{2.3}$$

Table 2 demonstrates a few standard distributions obtained from (2.1) by choosing appropriate values of parameters (p and α_j, $j = 0, 1, 2, \ldots$). The mathematical form of pdf, as given in (2.3), is very useful to derive the expressions for recurrence relations for single and product moments of generalized order statistics.

Notations

For $n = 1, 2, 3, \ldots, 1 \le r < s \le n,\ k \ge 1$ and $u, v \in \{0, 1, 2, \ldots\}$, we denote by

(a) $\mu^u_{r:n,m,k} = E(X(r,n,m,k))^u$ (2.4)

(b) $\mu^{u,v}_{r,s:n,m,k} = E[\{X(r,n,m,k)\}^u\{X(s,n,m,k)\}^v]$ (2.5)

(c) $\mu^u_{r:n,\widetilde{m},k} = E(X(r,n,\widetilde{m},k))^u$ (2.6)

(d) $\mu^{u,v}_{r,s:n,\widetilde{m},k} = E[\{X(r,n,\widetilde{m},k)\}^u\{X(s,n,\widetilde{m},k)\}^v]$ (2.7)

(e) $M_{r:n,m,k}(t) = E(e^{tX(r,n,m,k)})$ and $M_{r:n,\widetilde{m},k}(t) = E(e^{tX(r,n,\widetilde{m},k)})$ (2.8)

(f) $M^u_{r:n,m,k}(t) = \dfrac{\partial^u}{\partial t^u}[M_{r:n,m,k}(t)]$ and $M^u_{r:n,m,k}(0) = \mu^u_{r:n,m,k}$ (2.9)

(g) $M^u_{r:n,\widetilde{m},k}(t) = \dfrac{\partial^u}{\partial t^u}[M_{r:n,\widetilde{m},k}(t)]$ and $M^u_{r:n,\widetilde{m},k}(0) = \mu^u_{r:n,\widetilde{m},k}$ (2.10)

(h) $M_{r,s:n,m,k}(t_1,t_2) = E(e^{t_1X(r,n,m,k)+t_2X(s,n,m,k)})$ (2.11)

(i) $M_{r,s:n,\widetilde{m},k}(t_1,t_2) = E(e^{t_1X(r,n,\widetilde{m},k)+t_2X(s,n,\widetilde{m},k)})$ (2.12)

(j) $M^{u,v}_{r,s:n,m,k}(t_1,t_2) = \dfrac{\partial^{u+v}}{\partial t_1^u \partial t_2^v}[M_{r,s:n,m,k}(t_1,t_2)]$ (2.13)

(k) $M^{u,v}_{r,s:n,m,k}(0,0) = \mu^{u,v}_{r,s:n,m,k}$ (2.14)

(l) $M^{u,v}_{r,s:n,\widetilde{m},k}(t_1,t_2) = \dfrac{\partial^{u+v}}{\partial t_1^u \partial t_2^v}[M_{r,s:n,\widetilde{m},k}(t_1,t_2)]$ (2.15)

(m) $M^{u,v}_{r,s:n,\widetilde{m},k}(0,0) = \mu^{u,v}_{r,s:n,\widetilde{m},k}$ (2.16)

In this section we use the relation (2.3) to establish the recurrence relations for marginal and joint moment generating functions of generalized order statistics arising from pth order exponential distribution (2.1) for Case I and Case II.

Case I: $m_1 = m_2 = \ldots = m_{n-1} = m$.

Theorem 2.1. For $n = 1, 2, 3, \ldots, 1 \le r \le n$ and $k \ge 1$,

$$M_{r:n,m,k}(t) = \gamma_r \sum_{j=0}^{p}\sum_{i=0}^{j}(-1)^i \frac{j!}{(j-i)!}\alpha_j \left(\frac{M^{j-i}_{r:n,m,k} - M^{j-i}_{r-1:n,m,k}}{t^{i+1}}\right). \tag{2.17}$$

Proof: Using (1.3), the moment generating function of $X(r,n,m,k)$ is given by

$$M_{r:n,m,k}(t) = \frac{c_{r-1}}{(r-1)!}\int_0^\infty e^{tx}(\bar{F}(x))^{\gamma_r - 1} f(x) g_m^{r-1}(F(x))dx. \tag{2.18}$$

Substituting $f(x)$ from (2.3), we have

$$M_{r:n,m,k}(t) = \frac{c_{r-1}}{(r-1)!}\sum_{j=0}^{p}\int_0^\infty \alpha_j x^j e^{tx}(\bar{F}(x))^{\gamma_r} g_m^{r-1}(F(x))dx \tag{2.19}$$

$$= \frac{c_{r-1}}{(r-1)!}\sum_{j=0}^{p}\int_0^\infty \alpha_j D^j(e^{tx})(\bar{F}(x))^{\gamma_r} g_m^{r-1}(F(x))dx, \text{ where } D = \frac{\partial}{\partial t}$$

$$= \gamma_r \sum_{j=0}^{p}\alpha_j D^j\left[\frac{c_{r-2}}{(r-1)!}\int_0^\infty e^{tx}(\bar{F}(x))^{\gamma_r} g_m^{r-1}(F(x))dx\right] \tag{2.20}$$

Now by using (1.10) for $\omega(x) = e^{tx}$ with support $(0,\infty)$, we have

$$M_{r:n,m,k}(t) - M_{r-1:n,m,k}(t) = \frac{tc_{r-2}}{(r-1)!}\int_0^\infty e^{tx}(\bar{F}(x))^{\gamma_r} g_m^{r-1}(F(x))dx. \tag{2.21}$$

On combining relations (2.20) and (2.21), we have

$$M_{r:n,m,k}(t) = \gamma_r\sum_{j=0}^{p}\alpha_j D^j\left[\frac{M_{r:n,m,k}(t) - M_{r-1:n,m,k}(t)}{t}\right] \tag{2.22}$$

$$= \gamma_r\sum_{j=0}^{p}\sum_{i=0}^{j}\alpha_j\binom{j}{i}D^{j-i}(M_{r:n,m,k}(t) - M_{r-1:n,m,k}(t))D^i\left(\frac{1}{t}\right)$$

$$= \gamma_r\sum_{j=0}^{p}\sum_{i=0}^{j}(-1)^i\alpha_j\binom{j}{i}\left(\frac{M_{r:n,m,k}^{j-i} - M_{r-1:n,m,k}^{j-i}}{t^{i+1}}\right)i!, \tag{2.23}$$

which immediately leads to the recurrence relation as stated in (2.17).

Remark 2.1. After differentiating (2.22) with respect to t, u times, we get

$$M_{r:n,m,k}^u(t) = \gamma_r\sum_{j=0}^{p}\alpha_j D^{u+j}\left[\frac{M_{r:n,m,k}(t) - M_{r-1:n,m,k}(t)}{t}\right]$$

$$= \gamma_r\sum_{j=0}^{p}\sum_{i=0}^{u+j}\alpha_j\binom{u+j}{i}D^{u+j-i}(M_{r:n,m,k}(t) - M_{r-1:n,m,k}(t))D^i\left(\frac{1}{t}\right)$$

$$= \gamma_r\sum_{j=0}^{p}\alpha_j\sum_{i=0}^{u+j}(-1)^i\binom{u+j}{i}\left(\frac{M_{r:n,m,k}^{u+j-i}(t) - M_{r-1:n,m,k}^{u+j-i}(t)}{t^{i+1}}\right)i!. \tag{2.24}$$

Now,

$$\lim_{t\to 0} M_{r:n,m,k}^u(t) = \gamma_r\sum_{j=0}^{p}\alpha_j\left(\lim_{t\to 0}\sum_{i=0}^{u+j}(-1)^i\binom{u+j}{i}\left(\frac{M_{r:n,m,k}^{u+j-i}(t) - M_{r-1:n,m,k}^{u+j-i}(t)}{t^{i+1}}\right)i!\right). \tag{2.25}$$

The limiting expression appearing on the RHS of (2.25) is separately solved as

$$\begin{aligned}
&\lim_{t\to 0}\sum_{i=0}^{u+j}(-1)^i\binom{u+j}{i}\left(\frac{M_{r:n,m,k}^{u+j-i}(t)-M_{r-1:n,m,k}^{u+j-i}(t)}{t^{i+1}}\right)i! \to \frac{0}{0}\\
&=\sum_{i=0}^{u+j}(-1)^i\binom{u+j}{i}\left(\frac{\mu_{r:n,m,k}^{u+j+1}-\mu_{r-1:n,m,k}^{u+j+1}}{(i+1)!}\right)i! \text{ (using } L \text{ Hospital Rule)}\\
&=(\mu_{r:n,m,k}^{u+j+1}-\mu_{r-1:n,m,k}^{u+j+1})\sum_{i=0}^{u+j}(-1)^i\frac{1}{i+1}\binom{u+j}{i}\\
&=-\frac{\mu_{r:n,m,k}^{u+j+1}-\mu_{r-1:n,m,k}^{u+j+1}}{u+j+1}\sum_{i=0}^{u+j}(-1)^{i+1}\binom{u+j+1}{i+1}\\
&=\frac{\mu_{r:n,m,k}^{u+j+1}-\mu_{r-1:n,m,k}^{u+j+1}}{u+j+1}. \qquad (2.26)
\end{aligned}$$

On combining relations (2.25), (2.26) and (2.9), we have

$$\mu_{r:n,m,k}^{u}=\gamma_r\sum_{j=0}^{p}\frac{\alpha_j}{u+j+1}(\mu_{r:n,m,k}^{u+j+1}-\mu_{r-1:n,m,k}^{u+j+1}),$$

which is in agreement with Theorem 1 of Kumar (2010).

Theorem 2.2. For $n = 1, 2, 3, \ldots$, $1 \le r < s \le n$ and $k \ge 1$,

$$\begin{aligned}
M_{r,s:n,m,k}(t_1,t_2)=\gamma_s\sum_{j=0}^{p}\sum_{i=0}^{j}(-1)^i\frac{j!}{(j-i)!}\alpha_j\\
\times\left(\frac{M_{r,s:n,m,k}^{0,j-i}(t_1,t_2)-M_{r-1,s:n,m,k}^{0,j-i}(t_1,t_2)}{t_2^{i+1}}\right). \qquad (2.27)
\end{aligned}$$

Proof: Using (1.4), the joint moment generating function of $X(r,n,m,k)$ and $X(s,n,m,k)$ is given by

$$\begin{aligned}
M_{r,s:n,m,k}(t_1,t_2)=\frac{c_{s-1}}{(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty e^{t_1x+t_2y}(\bar F(x))^m f(x)g_m^{r-1}(F(x))\\
\times(h_m(F(y))-h_m(F(x)))^{s-r-1}(\bar F(y))^{\gamma_s-1}f(y)dydx. \qquad (2.28)
\end{aligned}$$

Substituting $f(y)$ from (2.3), we have

$$M_{r,s:n,m,k}(t_1,t_2) = \frac{c_{s-1}}{(r-1)!(s-r-1)!}\sum_{j=0}^{p}\alpha_j\int_0^\infty\int_x^\infty y^j e^{t_1x+t_2y}(\bar{F}(x))^m f(x) g_m^{r-1}(F(x))$$
$$\times (h_m(F(y)) - h_m(F(x)))^{s-r-1}(\bar{F}(y))^{\gamma_s} dydx$$
$$= \frac{c_{s-1}}{(r-1)!(s-r-1)!}\sum_{j=0}^{p}\alpha_j\int_0^\infty\int_x^\infty D^j e^{t_1x+t_2y}(\bar{F}(x))^m f(x) g_m^{r-1}(F(x))$$
$$\times (h_m(F(y)) - h_m(F(x)))^{s-r-1}(\bar{F}(y))^{\gamma_s} dydx, \tag{2.29}$$

where $D = \frac{\partial}{\partial t_2}$.

Now by using (1.12) for $\omega(x,y) = e^{t_1x+t_2y}$ with support $(0,\infty)$, we have

$$M_{r,s:n,m,k}(t_1,t_2) - M_{r,s-1:n,m,k}(t_1,t_2)$$
$$= \frac{t_2 c_{s-2}}{(r-1)!(s-r-1)!}\int_0^\infty\int_x^\infty e^{t_1x+t_2y}(\bar{F}(x))^m f(x) g_m^{r-1}(F(x))$$
$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}(\bar{F}(y))^{\gamma_s} dydx. \tag{2.30}$$

On combining (2.29) and (2.30), we have

$$M_{r,s:n,m,k}(t_1,t_2)$$
$$= \gamma_s\sum_{j=0}^{p}\alpha_j D^j\left(\frac{M_{r,s:n,m,k}(t_1,t_2) - M_{r,s-1:n,m,k}(t_1,t_2)}{t_2}\right) \tag{2.31}$$
$$= \gamma_s\sum_{j=0}^{p}\sum_{i=0}^{j}\alpha_j\binom{j}{i}D^{j-i}(M_{r,s:n,m,k}(t_1,t_2) - M_{r,s-1:n,m,k}(t_1,t_2))D^i\left(\frac{1}{t_2}\right)$$
$$= \gamma_s\sum_{j=0}^{p}\sum_{i=0}^{j}\alpha_j\binom{j}{i}\left(\frac{M^{0,j-i}_{r,s:n,m,k}(t_1,t_2) - M^{0,j-i}_{r,s-1:n,m,k}(t_1,t_2)}{t_2^{i+1}}\right)(-1)^i i!,$$

which on simplification leads to the recurrence relation as stated in (2.27).

Remark 2.2. After differentiating both sides of (2.31) with respect to t_1, u times, we get

$$M^{u,0}_{r,s:n,m,k}(t_1,t_2) = \gamma_s\sum_{j=0}^{p}\alpha_j D^j\left(\frac{M^{u,0}_{r,s:n,m,k}(t_1,t_2) - M^{u,0}_{r,s-1:n,m,k}(t_1,t_2)}{t_2}\right). \tag{2.32}$$

Further, differentiating (2.31) with respect to t_2, v times, we get

$$M^{u,v}_{r,s:n,m,k}(t_1,t_2) = \gamma_s \sum_{j=0}^{p} \alpha_j D^{v+j} \left(\frac{M^{u,0}_{r,s:n,m,k}(t_1,t_2) - M^{u,0}_{r,s-1:n,m,k}(t_1,t_2)}{t_2} \right) \quad (2.33)$$

$$= \gamma_s \sum_{j=0}^{p} \alpha_j \sum_{i=0}^{v+j} (-1)^i i! \binom{v+j}{i} \left(\frac{M^{u,v+j-i}_{r,s:n,m,k}(t_1,t_2) - M^{u,v+j-i}_{r,s-1:n,m,k}(t_1,t_2)}{t_2^{i+1}} \right).$$

Putting $t_1 = 0$, we have

$$M^{u,v}_{r,s:n,m,k}(0,t_2) = \gamma_s \sum_{j=0}^{p} \alpha_j \sum_{i=0}^{v+j} (-1)^i i! \binom{v+j}{i} \left(\frac{M^{u,v+j-i}_{r,s:n,m,k}(0,t_2) - M^{u,v+j-i}_{r,s-1:n,m,k}(0,t_2)}{t_2^{i+1}} \right). \quad (2.34)$$

Now,

$$\lim_{t_2 \to 0} M^{u,v}_{r,s:n,m,k}(0,t_2)$$

$$= \gamma_s \sum_{j=0}^{p} \alpha_j \left[\lim_{t_2 \to 0} \sum_{i=0}^{v+j} (-1)^i i! \binom{v+j}{i} \left(\frac{M^{u,v+j-i}_{r,s:n,m,k}(0,t_2) - M^{u,v+j-i}_{r,s-1:n,m,k}(0,t_2)}{t_2^{i+1}} \right) \right] \to \frac{0}{0}, \quad (2.35)$$

which on using (2.14) and L' Hospital's Rule leads to

$$\mu^{u,v}_{r,s:n,m,k} = \gamma_s \sum_{j=0}^{p} \alpha_j \left[\sum_{i=0}^{v+j} (-1)^i i! \binom{v+j}{i} \frac{\mu^{u,v+j+1}_{r,s:n,m,k} - \mu^{u,v+j+1}_{r,s-1:n,m,k}}{(i+1)!} \right]$$

$$= \gamma_s \sum_{j=0}^{p} \alpha_j (\mu^{u,v+j+1}_{r,s:n,m,k} - \mu^{u,v+j+1}_{r,s-1:n,m,k}) \sum_{i=0}^{v+j} (-1)^i \frac{1}{i+1} \binom{v+j}{i}$$

$$= -\gamma_s \sum_{j=0}^{p} \alpha_j \left(\frac{\mu^{u,v+j+1}_{r,s:n,m,k} - \mu^{u,v+j+1}_{r,s-1:n,m,k}}{v+j+1} \right) \sum_{i=0}^{v+j} (-1)^{i+1} \binom{v+j+1}{i+1}$$

$$= \gamma_s \sum_{j=0}^{p} \frac{\alpha_j}{v+j+1} (\mu^{u,v+j+1}_{r,s:n,m,k} - \mu^{u,v+j+1}_{r,s-1:n,m,k}), \quad (2.36)$$

which is in agreement with Theorem 3 of Kumar (2010).

Case II: $\gamma_i \neq \gamma_j$; $i \neq j$, $i, j = 1, 2, \ldots, n-1$.

Theorem 2.3. For $n = 1, 2, 3, \ldots$, $1 \le r \le n$ and $k \ge 1$,

$$M_{r:n,\tilde{m},k}(t) = \gamma_r \sum_{j=0}^{p} \sum_{i=0}^{j} (-1)^i \frac{j!}{(j-i)!} \alpha_j \left(\frac{M^{j-i}_{r:n,\tilde{m},k} - M^{j-i}_{r-1:n,\tilde{m},k}}{t^{i+1}} \right). \quad (2.37)$$

Proof: On employing (1.6), the moment generating function of $X(r, n, \widetilde{m}, k)$ is given by

$$M_{r:n,\widetilde{m},k}(t) = c_{r-1} \int_0^\infty e^{tx} \sum_{i=1}^{r} a_i(r)(\bar{F}(x))^{\gamma_i - 1} f(x)dx. \tag{2.38}$$

Substituting $f(x)$ from (2.3), we have

$$\begin{aligned} M_{r:n,\widetilde{m},k}(t) &= c_{r-1} \sum_{j=0}^{p} \sum_{i=1}^{r} \alpha_j a_i(r) \int_0^\infty x^j e^{tx} (\bar{F}(x))^{\gamma_i} dx \\ &= \gamma_r \sum_{j=0}^{p} \alpha_j D^j \left[c_{r-2} \sum_{i=1}^{r} a_i(r) \int_0^\infty (e^{tx})(\bar{F}(x))^{\gamma_i} dx \right], \text{ where } D = \frac{\partial}{\partial t}. \end{aligned} \tag{2.39}$$

On using (1.11) for $\omega(x) = e^{tx}$ with support $(0, \infty)$, we have

$$\begin{aligned} M_{r:n,\widetilde{m},k}(t) &= \gamma_r \sum_{j=0}^{p} \alpha_j D^j \left[\frac{M_{r:n,\widetilde{m},k}(t) - M_{r-1:n,\widetilde{m},k}(t)}{t} \right] \qquad (2.40) \\ &= \gamma_r \sum_{j=0}^{p} \sum_{i=0}^{j} \alpha_j \binom{j}{i} D^{j-i} (M_{r:n,\widetilde{m},k}(t) - M_{r-1:n,\widetilde{m},k}(t)) D^i \left(\frac{1}{t} \right) \\ &= \gamma_r \sum_{j=0}^{p} \sum_{i=0}^{j} (-1)^i \alpha_j \binom{j}{i} \left(\frac{M^{j-i}_{r:n,\widetilde{m},k} - M^{j-i}_{r-1:n,\widetilde{m},k}}{t^{i+1}} \right) i!, \end{aligned}$$

which on little simplification proves the recurrence relation given in (2.37).

Remark 2.3. After differentiating (2.40) both sides with respect to t, u times, we get

$$\begin{aligned} M^u_{r:n,\widetilde{m},k}(t) &= \gamma_r \sum_{j=0}^{p} \alpha_j D^{u+j} \left[\frac{M_{r:n,\widetilde{m},k}(t) - M_{r-1:n,\widetilde{m},k}(t)}{t} \right] \\ &= \gamma_r \sum_{j=0}^{p} \sum_{i=0}^{u+j} \alpha_j \binom{u+j}{i} D^{u+j-i} (M_{r:n,\widetilde{m},k}(t) - M_{r-1:n,\widetilde{m},k}(t)) D^i \left(\frac{1}{t} \right) \\ &= \gamma_r \sum_{j=0}^{p} \sum_{i=0}^{u+j} (-1)^i \alpha_j \binom{u+j}{i} \left(\frac{M^{u+j-i}_{r:n,\widetilde{m},k}(t) - M^{u+j-i}_{r-1:n,\widetilde{m},k}(t)}{t^{i+1}} \right) i!. \end{aligned}$$

Now,

$$\lim_{t\to 0} M^{u}_{r:n,\widetilde{m},k}(t) = \gamma_r \sum_{j=0}^{p} \alpha_j \left[\lim_{t\to 0} \sum_{i=0}^{u+j} (-1)^i \binom{u+j}{i} \left(\frac{M^{u+j-i}_{r:n,\widetilde{m},k}(t) - M^{u+j-i}_{r-1:n,\widetilde{m},k}(t)}{t^{i+1}} \right) i! \right]. \tag{2.41}$$

It may be noted that the expression inside the square brackets on the RHS of (2.41) $\to \frac{0}{0}$ form. Thus, on using (2.10) and L' Hospital's Rule, we get

$$\begin{aligned}
\mu^{u}_{r:n,\widetilde{m},k} &= \gamma_r \sum_{j=0}^{p} \alpha_j \left[\sum_{i=0}^{u+j} (-1)^i \binom{u+j}{i} \left(\frac{\mu^{u+j+1}_{r:n,\widetilde{m},k} - \mu^{u+j+1}_{r-1:n,\widetilde{m},k}}{(i+1)!} \right) i! \right] \\
&= \gamma_r \sum_{j=0}^{p} \alpha_j \left[(\mu^{u+j+1}_{r:n,\widetilde{m},k} - \mu^{u+j+1}_{r-1:n,\widetilde{m},k}) \sum_{i=0}^{u+j} (-1)^i \frac{1}{i+1} \binom{u+j}{i} \right] \\
&= \gamma_r \sum_{j=0}^{p} \alpha_j \left[-\frac{\mu^{u+j+1}_{r:n,\widetilde{m},k} - \mu^{u+j+1}_{r-1:n,\widetilde{m},k}}{u+j+1} \sum_{i=0}^{u+j} (-1)^{i+1} \binom{u+j+1}{i+1} \right] \\
&= \gamma_r \sum_{j=0}^{p} \frac{\alpha_j}{u+j+1} (\mu^{u+j+1}_{r:n,\widetilde{m},k} - \mu^{u+j+1}_{r-1:n,\widetilde{m},k}),
\end{aligned} \tag{2.42}$$

which is in agreement with Theorem 2 of Kumar (2010). It may be mentioned that there is a printing error in the result given in Theorem 2 of Kumar (2010).

Theorem 2.4. For $n = 1, 2, 3, \ldots$, $1 \le r < s \le n$ and $k \ge 1$,

$$\begin{aligned}
&M_{r,s:n,\widetilde{m},k}(t_1, t_2) \\
&= \gamma_s \sum_{j=0}^{p} \sum_{i=0}^{j} \alpha_j (-1)^i \frac{j!}{(j-i)!} \left[\frac{M^{0,j-i}_{r,s:n,\widetilde{m},k}(t_1,t_2) - M^{0,j-i}_{r,s-1:n,\widetilde{m},k}(t_1,t_2)}{t_2^{i+1}} \right].
\end{aligned} \tag{2.43}$$

Proof: On using (1.7), we can write

$$\begin{aligned}
M_{r,s:n,\widetilde{m},k}(t_1, t_2) = c_{s-1} \int_0^\infty \int_x^\infty e^{t_1 x + t_2 y} \left[\sum_{i=1}^{r} a_i(r) (\bar{F}(x))^{\gamma_i} \right] \frac{f(x)}{\bar{F}(x)} \\
\times \left[\sum_{i=r+1}^{s} a_i^r(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)} \right)^{\gamma_i} \right] \frac{f(y)}{\bar{F}(y)} dy dx.
\end{aligned} \tag{2.44}$$

Using (2.3) and putting $\dfrac{f(y)}{\bar{F}(y)} = \sum\limits_{j=0}^{p} \alpha_j y^j$ in (2.44), we get

$$M_{r,s:n,\widetilde{m},k}(t_1,t_2) = c_{s-1} \sum_{j=0}^{p} \alpha_j \int_0^\infty \int_x^\infty y^j e^{t_1 x + t_2 y} \left[\sum_{i=1}^{r} a_i(r) (\bar{F}(x))^{\gamma_i} \right] \frac{f(x)}{\bar{F}(x)} \times \left[\sum_{i=r+1}^{s} a_i^r(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)} \right)^{\gamma_i} \right] dydx \tag{2.45}$$

$$= \gamma_s \sum_{j=0}^{p} \alpha_j D^j \left[c_{s-2} \int_0^\infty \int_x^\infty e^{t_1 x + t_2 y} \left[\sum_{i=1}^{r} a_i(r) (\bar{F}(x))^{\gamma_i} \right] \times \frac{f(x)}{\bar{F}(x)} \left[\sum_{i=r+1}^{s} a_i^r(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)} \right)^{\gamma_i} \right] dydx \right], \text{ where } D = \frac{\partial}{\partial t_2}. \tag{2.46}$$

On using (1.13) for $\omega(x,y) = e^{t_1 x + t_2 y}$ with support $(0,\infty)$, we get

$$M_{r,s:n,\widetilde{m},k}(t_1,t_2) = \gamma_s \sum_{j=0}^{p} \alpha_j D^j \left[\frac{M_{r,s:n,\widetilde{m},k}(t_1,t_2) - M_{r,s-1:n,\widetilde{m},k}(t_1,t_2)}{t_2} \right] \tag{2.47}$$

$$= \gamma_s \sum_{j=0}^{p} \sum_{i=0}^{j} \alpha_j \binom{j}{i} D^{j-i} [M_{r,s:n,\widetilde{m},k}(t_1,t_2) - M_{r,s-1:n,\widetilde{m},k}(t_1,t_2)] D^i \left(\frac{1}{t_2} \right)$$

$$= \gamma_s \sum_{j=0}^{p} \sum_{i=0}^{j} \alpha_j (-1)^i \binom{j}{i} \left[\frac{M^{0,j-i}_{r,s:n,\widetilde{m},k}(t_1,t_2) - M^{0,j-i}_{r,s-1:n,\widetilde{m},k}(t_1,t_2)}{t_2^{i+1}} \right] i!, \tag{2.48}$$

which establishes the recurrence relation given in (2.43).

Remark 2.4. After differentiating (2.47) both sides with respect to t_1, u times, and with respect to t_2, v times and then putting $t_1 = t_2 = 0$ and proceeding similarly as in Remark 2.2, we shall derive the recurrence relation:

$$\mu^{u,v}_{r,s:n,\widetilde{m},k} = \gamma_s \sum_{j=0}^{p} \frac{\alpha_j}{v+j+1} (\mu^{u,v+j+1}_{r,s:n,\widetilde{m},k} - \mu^{u,v+j+1}_{r,s-1:n,\widetilde{m},k}), \tag{2.49}$$

which is similar to Theorem 4 of Kumar (2010). It may be mentioned that there is a mistake in the result given in Theorem 4 of Kumar (2010).

3 Recurrence relations for MGF's of generalized order statistics from LTLD

A random variable X is said to have Left Truncated Logistic Distribution (LTLD) if its probability density function is given by

$$f(x) = \frac{(e^{\beta}+1)e^{-x}}{(e^{-x}+1)^2}, \quad \beta \leq x < \infty, -\infty < \beta < \infty, \tag{3.1}$$

and its cumulative distribution function is given by

$$F(x) = \frac{e^{\beta}(e^{-\beta} - e^{-x})}{e^{-x}+1}. \tag{3.2}$$

The characterizing differential equation for LTLD is given by

$$\begin{aligned} \frac{1-F(x)}{f(x)} &= 1 + e^{-x} \\ &= \sum_{j=0}^{\infty} \alpha_j x^j, \text{ where } \alpha_j = \begin{cases} 2, & j = 0, \\ \dfrac{(-1)^j}{j!}, & j \geq 1, \end{cases} \end{aligned} \tag{3.3}$$

which plays an important role in deriving the recurrence relations for single and product moments of generalized order statistics arising from left truncated logistic distribution (3.1).

In this section, we shall derive the recurrence relations for marginal and joint moment generating functions of generalized order statistics from left truncated logistic distribution given in (3.1).

Case I: $m_1 = m_2 = \ldots = m_{n-1} = m$.

Theorem 3.1. Fix a positive integer k. For $n \in N$, $m \in Z$ and $1 \leq r \leq n$,

$$M_{r:n,m,k}(t) = M_{r-1:n,m,k}(t) + \frac{t}{\gamma_r} \sum_{j=0}^{\infty} \alpha_j M^{j}_{r:n,m,k}. \tag{3.4}$$

Proof: Using (1.3), the moment generating function of $X(r,n,m,k)$ from the left truncated logistic distribution (3.1) is given by

$$M_{r:n,m,k}(t) = \frac{c_{r-1}}{(r-1)!} \int_{\beta}^{\infty} e^{tx} (\bar{F}(x))^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) dx. \tag{3.5}$$

Substituting $\omega(x) = e^{tx}$ in (1.10) and using (3.3), we have

$$M_{r:n,m,k}(t) - M_{r-1:n,m,k}(t)$$
$$= \frac{tc_{r-1}}{\gamma_r(r-1)!}\int_\beta^\infty e^{tx}(\bar{F}(x))^{\gamma_r-1}\left(\sum_{j=0}^{\infty}\alpha_j x^j\right) f(x)g_m^{r-1}(F(x))dx, \quad (3.6)$$

which after simplification leads to (3.4).

Remark 3.1. After differentiating both sides of (3.4), with respect to t, $u+1$ times and then putting $t = 0$, we shall derive the recurrence relation among single moments of generalized order statistics arising from left truncated logistic distribution as follows:

$$\mu_{r:n,m,k}^{u+1} = \mu_{r-1:n,m,k}^{u+1} + \frac{u+1}{\gamma_r}\sum_{j=0}^{\infty}\alpha_j \mu_{r:n,m,k}^{u+j}. \quad (3.7)$$

Theorem 3.2. Fix a positive integer k. For $n \in N$, $m \in Z$ and $1 \le r < s \le n$,

$$M_{r,s:n,m,k}(t_1,t_2) = M_{r,s-1:n,m,k}(t_1,t_2) + \frac{t_2}{\gamma_s}\sum_{j=0}^{\infty}\alpha_j M_{r,s:n,m,k}^{0,j}(t_1,t_2). \quad (3.8)$$

Proof: Using (1.4), the expression for joint moment generating function of $X(r,n,m,k)$ and $X(s,n,m,k)$ is:

$$M_{r,s:n,m,k}(t_1,t_2) = \frac{c_{s-1}}{(r-1)!(s-r-1)!}\int_\beta^\infty\int_x^\infty e^{t_1x+t_2y}(\bar{F}(x))^m f(x)g_m^{r-1}(F(x))$$
$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1}(\bar{F}(y))^{\gamma_s-1}f(y)dydx. \quad (3.9)$$

Substituting $\omega(x,y) = e^{t_1x+t_2y}$ in (1.12), we have

$$M_{r,s:n,m,k}(t_1,t_2) - M_{r,s-1:n,m,k}(t_1,t_2)$$
$$= \frac{t_2c_{s-2}}{(r-1)!(s-r-1)!}\int_\beta^\infty\int_x^\infty e^{t_1x+t_2y}(\bar{F}(x))^m$$
$$\times f(x)g_m^{r-1}(F(x))[h_m(F(y)) - h_m(F(x))]^{s-r-1}(\bar{F}(y))^{\gamma_s}dydx. \quad (3.10)$$

Further, on using (3.3) in (3.10), we get

$$M_{r,s:n,m,k}(t_1,t_2) - M_{r,s-1:n,m,k}(t_1,t_2)$$
$$= \frac{t_2 c_{s-1}}{\gamma_s (r-1)!(s-r-1)!} \int_\alpha^\infty \int_x^\infty e^{t_1 x + t_2 y} \left(\sum_{j=0}^{\infty} \alpha_j y^j \right) (\bar{F}(x))^m f(x) g_m^{r-1}(F(x))$$
$$\times [h_m(F(y)) - h_m(F(x))]^{s-r-1} (\bar{F}(y))^{\gamma_s - 1} f(y) dy dx, \tag{3.11}$$

which on simplification and using (3.9) takes the form as stated in (3.8).

Remark 3.2. After differentiating both sides of (3.8) with respect to t_1, u times and with respect to t_2, $v+1$ times and then putting $t_1 = t_2 = 0$, we shall derive the recurrence relation for product moments of generalized order statistics arising from left truncated logistic distribution as follows:

$$\mu_{r,s:n,m,k}^{u,v+1} = \mu_{r,s-1:n,m,k}^{u,v+1} + \frac{v+1}{\gamma_s} \sum_{j=0}^{\infty} \alpha_j \mu_{r,s:n,m,k}^{u,v+j}. \tag{3.12}$$

Case II: $\gamma_i \neq \gamma_j$; $i \neq j$, $i, j = 1, 2, \ldots, n-1$.

Theorem 3.3. Fix a positive integer k. For $n \in N$, $m \in Z$ and $1 \leq r \leq n$,

$$M_{r:n,\widetilde{m},k}(t) = M_{r-1:n,\widetilde{m},k}(t) + \frac{t}{\gamma_r} \sum_{j=0}^{\infty} \alpha_j M_{r:n,\widetilde{m},k}^{j}(t). \tag{3.13}$$

Proof: On using (1.6), the moment generating function of $X(r, n, \widetilde{m}, k)$ is given by

$$M_{r:n,\widetilde{m},k}(t) = c_{r-1} \int_\beta^\infty e^{tx} \sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i - 1} f(x) dx. \tag{3.14}$$

Substituting $\omega(x) = e^{tx}$ in (1.11) and using (3.3), we have

$$M_{r:n,\widetilde{m},k}(t) - M_{r-1:n,\widetilde{m},k}(t)$$
$$= \frac{t c_{r-1}}{\gamma_r} \int_\beta^\infty e^{tx} \left(\sum_{i=1}^{r} a_i(r)(1-F(x))^{\gamma_i - 1} \right) \left(\sum_{j=0}^{\infty} \alpha_j x^j \right) f(x) dx, \tag{3.15}$$

which on further simplification and using (3.14) leads to (3.13).

Remark 3.3. After differentiating both sides of (3.13), with respect to t, $u+1$ times and then putting $t = 0$, we shall derive the recurrence relation for single moments of generalized order statistics from left truncated logistic distribution as follows:

$$\mu^{u+1}_{r:n,\widetilde{m},k} = \mu^{u+1}_{r-1:n,\widetilde{m},k} + \frac{u+1}{\gamma_r}\sum_{j=0}^{\infty}\alpha_j \mu^{u+j}_{r:n,\widetilde{m},k}. \tag{3.16}$$

Theorem 3.4. Fix a positive integer k. For $n = 1, 2, 3, \ldots$, $1 \le r < s \le n$ and $k \ge 1$,

$$M_{r,s:n,\widetilde{m},k}(t_1,t_2) = M_{r,s-1:n,\widetilde{m},k}(t_1,t_2) + \frac{t_2}{\gamma_s}\sum_{j=0}^{\infty}\alpha_j M^{0,j}_{r,s:n,\widetilde{m},k}(t_1,t_2). \tag{3.17}$$

Proof: From (1.7), the joint moment generating function of $X(r,n,\widetilde{m},k)$ and $X(s,n,\widetilde{m},k)$ is given by

$$\begin{aligned} M_{r,s:n,\widetilde{m},k}(t_1,t_2) = c_{s-1}\int_{\beta}^{\infty}\int_{x}^{\infty} e^{t_1x+t_2y}\left[\sum_{i=1}^{r} a_i(r)(\bar{F}(x))^{\gamma_i}\right]\frac{f(x)}{\bar{F}(x)} \\ \times\left[\sum_{i=r+1}^{s} a_i^r(s)\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i}\right]\frac{f(y)}{\bar{F}(y)}dydx. \end{aligned} \tag{3.18}$$

Substituting $\omega(x,y) = e^{t_1x+t_2y}$ in (1.13) and using (3.3), we have

$$\begin{aligned} &M_{r,s:n,\widetilde{m},k}(t_1,t_2) - M_{r,s-1:n,\widetilde{m},k}(t_1,t_2) \\ &= \frac{t_2c_{s-1}}{\gamma_s}\int_{\beta}^{\infty}\int_{x}^{\infty} e^{t_1x+t_2y}\left[\sum_{i=1}^{r} a_i(r)(\bar{F}(x))^{\gamma_i}\right]\frac{f(x)}{\bar{F}(x)} \\ &\times\left[\sum_{i=r+1}^{s} a_i^r(s)\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i}\right]\left(\sum_{j=0}^{\infty}\alpha_j y^j\right)\frac{f(y)}{\bar{F}(y)}dydx. \end{aligned} \tag{3.19}$$

On further simplification of (3.19) and using (3.18), it leads to the recurrence relation as stated in (3.17).

Remark 3.4. After differentiating both sides of (3.17), with respect to t_1, u times and with respect to t_2, $v+1$ times and then putting $t_1 = t_2 = 0$, we shall derive the recurrence relation for the product moments of generalized order statistics from left truncated logistic distribution as follows:

$$\mu^{u,v+1}_{r,s:n,\widetilde{m},k} = \mu^{u,v+1}_{r,s-1:n,\widetilde{m},k} + \frac{v+1}{\gamma_s}\sum_{j=0}^{\infty}\alpha_j \mu^{u,v+j}_{r,s:n,\widetilde{m},k}. \tag{3.20}$$

4 Recurrence relations for MGF's of generalized order statistics from EVD

A random variable X is said to have the standard extreme value distribution (EVD) if its probability density function is of the form

$$f(x) = \alpha e^{\alpha x} \times e^{-e^{\alpha x}}, \quad -\infty < x < \infty, \alpha > 0 \tag{4.1}$$

and the cumulative distribution function is given by

$$F(x) = 1 - e^{-e^{\alpha x}}, \quad -\infty < x < \infty, \alpha > 0. \tag{4.2}$$

The characterizing differential equation for extreme value distribution (4.1) is given by

$$f(x) = \left(\sum_{j=0}^{\infty} \frac{\alpha^{j+1} x^j}{j!} \right) (1 - F(x)). \tag{4.3}$$

On comparing (4.3) with (2.3) it may be noted that the recurrence relations for marginal and joint moment generating functions of generalized order statistics from extreme value distribution with pdf (4.1) can easily be written down from the corresponding results of pth order exponential distribution given in Theorems 2.1-2.4 by substituting therein $p = \infty$ and $\alpha_j = \frac{\alpha^{j+1}}{j!}$, and the same are presented in this section as Theorem 4.1.

Theorem 4.1. For $n = 1, 2, 3, \ldots$, $1 \le r < s \le n$ and $k \ge 1$,

Case I: $m_1 = m_2 = \ldots = m_{n-1} = m$.

(a) $$M_{r:n,m,k}(t) = \gamma_r \sum_{j=0}^{\infty} \sum_{i=0}^{j} (-1)^i \frac{\alpha^{j+1}}{(j-i)!} \left(\frac{M^{j-i}_{r:n,m,k} - M^{j-i}_{r-1:n,m,k}}{t^{i+1}} \right) \tag{4.4}$$

(b) $$M_{r,s:n,m,k}(t_1, t_2) = \gamma_s \sum_{j=0}^{\infty} \sum_{i=0}^{j} (-1)^i \frac{\alpha^{j+1}}{(j-i)!} \times \left(\frac{M^{0,j-i}_{r,s:n,m,k}(t_1, t_2) - M^{0,j-i}_{r-1,s:n,m,k}(t_1, t_2)}{t_2^{i+1}} \right) \tag{4.5}$$

Case II: $\gamma_i \neq \gamma_j$, $i \neq j$, $i, j = 1, 2, \ldots, n-1$.

$$\text{(c)} \quad M_{r:n,\widetilde{m},k}(t) = \gamma_r \sum_{j=0}^{\infty} \sum_{i=0}^{j} (-1)^i \frac{\alpha^{j+1}}{(j-i)!} \left(\frac{M_{r:n,\widetilde{m},k}^{j-i} - M_{r-1:n,\widetilde{m},k}^{j-i}}{t^{i+1}} \right) \tag{4.6}$$

$$\text{(d)} \quad M_{r,s:n,\widetilde{m},k}(t_1, t_2) = \gamma_s \sum_{j=0}^{\infty} \sum_{i=0}^{j} (-1)^i \frac{\alpha^{j+1}}{(j-i)!} \times \left(\frac{M_{r,s:n,\widetilde{m},k}^{0,j-i}(t_1, t_2) - M_{r,s-1:n,\widetilde{m},k}^{0,j-i}(t_1, t_2)}{t_2^{i+1}} \right). \tag{4.7}$$

Remark 4.1. Differentiating (4.4) and (4.6) with respect to t, u times and then putting $t = 0$, and similarly, differentiating (4.5) and (4.7) with respect to t_1, u times and with respect to t_2, v times and then putting $t_1 = t_2 = 0$, we get the recurrence relations for the single and product moments of generalized order statistics arising from extreme value distribution (4.1) for Case I and Case II, which are given below:

Case I: $m_1 = m_2 = \ldots = m_{n-1} = m$.

$$\text{(a)} \quad \mu_{r:n,m,k}^{u} = \gamma_r \sum_{j=0}^{\infty} \frac{\alpha^{j+1}}{j!(u+j+1)} (\mu_{r:n,m,k}^{u+j+1} - \mu_{r-1:n,m,k}^{u+j+1}) \tag{4.8}$$

$$\text{(b)} \quad \mu_{r,s:n,m,k}^{u,v} = \gamma_s \sum_{j=0}^{\infty} \frac{\alpha^{j+1}}{j!(v+j+1)} (\mu_{r,s:n,m,k}^{u,v+j+1} - \mu_{r,s-1:n,m,k}^{u,v+j+1}) \tag{4.9}$$

Case II: $\gamma_i \neq \gamma_j$, $i \neq j$, $i, j = 1, 2, \ldots, n-1$.

$$\text{(c)} \quad \mu_{r:n,\widetilde{m},k}^{u} = \gamma_r \sum_{j=0}^{\infty} \frac{\alpha^{j+1}}{j!(u+j+1)} (\mu_{r:n,\widetilde{m},k}^{u+j+1} - \mu_{r-1:n,\widetilde{m},k}^{u+j+1}) \tag{4.10}$$

$$\text{(d)} \quad \mu_{r,s:n,\widetilde{m},k}^{u,v} = \gamma_s \sum_{j=0}^{\infty} \frac{\alpha^{j+1}}{j!(v+j+1)} (\mu_{r,s:n,\widetilde{m},k}^{u,v+j+1} - \mu_{r,s-1:n,\widetilde{m},k}^{u,v+j+1}). \tag{4.11}$$

5 Conclusion

In the study presented above, we demonstrate the recurrence relations for marginal and joint moment generating functions of generalized order statistics from p$^{\text{th}}$ order exponential distribution, left truncated logistic distribution and extreme value distribution. In all these distributions, the corresponding recurrence relations for single and product moments of generalized order statistics have also been deduced as special cases. The results so obtained are generalized versions of some of the recurrence relations obtained by Kumar (2010) and Saran and Pandey (2004, 2009).

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Marginal Laplace and Linnik Processes

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ABSTRACT

Kozubowski et al. (2005) introduced and studied a class of multivariate distributions called operator geometric stable (OGS) laws by generalizing operator stable and geometric stable laws and as a special case they studied marginal Laplace and Linnik (MLL) distributions. In this paper, corresponding to MLL distributions time series models with autoregressive structure is developed and some properties of the process are discussed.

Key words and Phrases: Autoregressive process, Geometric stable law, Laplace distribution, Linnik distribution, Operator geometric stable law.

1 Introduction

Let $\{X_n, n \geq 1\}$be a sequence of independent and identically distributed random vectors in R^d and let N_p be a geometric random variable with mean $\frac{1}{p}$, and independent of X_n's for each $p \in (0,1)$. If there exists a weak limit of $S_p = a(p)\sum_{i=1}^{N_p}(X_i + b(p))$, when $p \to 0$, where $a(p)$ is a linear operator on R^d and $b(p) \in \mathrm{R}^d$, then the class of limiting distributions is called operator geometric stable (OGS) laws. The OGS laws reduced to geometric stable laws under scalar normalization and in this case we obtain skew Laplace distribution as limiting law when components have finite second moment. If the sum S_p is deterministic with $N_p = n$, then the limiting distributions are operator stable laws and stable laws under scalar normalization. The class of OGS laws is a general class of distributions and developed by combining the concepts operator norming and geometric randomized sums. Kozubowski *et al.* (2005) studied this class of distributions in detail and derived many important properties. They discussed the one to one correspondence between operator stable and OGS distributions and established that the characteristic function $\Psi(t)$ of an OGS distribution has the form $\Psi(t) = \frac{1}{1 - \log\Theta(t)}, t \in \mathrm{R}^d$, where $\Theta(t)$ is the characteristic function of an operator stable distribution.

Let $X = (X_1, X_2)$ is a bivariate operator stable random vector with characteristic function

$$\Theta(t, s) = \exp(-\sigma^2 t^2 - \eta^\alpha |s|^\alpha), (t, s) \in \mathrm{R}^2, \alpha \in (0, 2]. \tag{1.1}$$

Then the corresponding OGS random vector $Y = (Y_1, Y_2)$ has the characteristic function

$$\Psi(t, s) = \frac{1}{1 + \sigma^2 t^2 + \eta^\alpha |s|^\alpha}, (t, s) \in \mathrm{R}^2, \alpha \in (0, 2]. \tag{1.2}$$

From this representation it is clear that the marginal distributions of $Y = (Y_1, Y_2)$ are Laplace and Linnik distributions with respective characteristic functions $\Psi(t, 0) = \frac{1}{1 + \sigma^2 t^2}$ and $\Psi(0, s) = \frac{1}{1 + \eta^\alpha |s|^\alpha}$, where $t \in \mathrm{R}, s \in \mathrm{R}$ and $\alpha \in (0, 2]$. Hence they referred the distribution of bivariate vector $Y = (Y_1, Y_2)$ with characteristic function (1.2) as an OGS law with marginal Laplace and Linnik distribution with parameters α, σ and η and denoted it as $Y \underline{\underline{d}} MLL_\alpha(\sigma, \eta)$.

We can observe that heavy tailed bivariate distribution with different tail behavior are useful for modeling bivariate data where the vectors are sums of a random number of small random shocks with heavy tails, and each component has a different tail index. Kozubowski *et al.* (2005) considered a bivariate data set on exchange log- rates for the US Dollar versus the German Deutsch Mark and the Japanese Yen. They established that marginal Laplace and Linnik distribution provides a better fit to this data. Despite the wide spread applications of this class of distributions in mathematical finance and other fields, their applications in time series modeling have not explored. In this paper we take up this task, in particular we develop and study a time series model using marginally geometric stable distributions.

The paper is organized as follows. In section 2, we study some properties and characterizations of marginal Laplace and Linnik distributions and present a first order autoregressive marginal Laplace and Linnik process. Also, we discuss some properties of the process, such as autocorrelation, asymptotic behavior etc. In section 3, we consider a generalized class of distributions, namely marginal asymmetric Laplace and asymmetric Linnik distribution by introducing asymmetric component in the marginal distributions. Corresponding time series model is developed in this section. In section 4, we consider a special case of operator υ- stable distribution by using gamma compounding, namely generalized marginal asymmetric Laplace and asymmetric Linnik distributions, and obtain time series model corresponding to this class of distributions.

2 Marginal Laplace and Linnik distribution and processes

2.1 Some properties and characterizations of the distribution

Kozubowski *et al.* (2005) derived the representation of $Y \underline{\underline{d}}\, MLL_\alpha(\sigma, \eta)$ as $Y \underline{\underline{d}}\, (Z^{1/2}X_1, Z^{1/\alpha}X_2)$, where Z is unit exponential, X_1,X_2 are normal and α-stable random variables with $\Theta(t, 0) = \exp(-\sigma^2 t^2)$ and $\Theta(0, s) = \exp(-\eta^\alpha |s|^\alpha)$ as respective characteristic functions. It may be noted that Z, $X_1$and X_2 are mutually independent random variables. Using this representation they derived the distribution function and density function of $Y = (Y_1, Y_2)$ and are given by

$$F(y_1, y_2) = \int_0^\infty \Phi(\frac{y_1}{\sqrt{2z}\,\sigma})\ \Omega_\alpha(\frac{y_2}{z^{1/\alpha}\,\eta})\ e^{-z} dz, \quad (y_1, y_2) \in \mathrm{R}^2 \tag{2.1}$$

and

$$f(y_1,y_2)=\frac{1}{2\sigma\,\eta\,\sqrt{\pi}}\int_0^{\infty} z^{-\left(\frac{1}{\alpha}+\frac{1}{2}\right)}\,e^{-z-\frac{y_1^2}{4\sigma^2 z}}\,P_\alpha\left(\frac{y_2}{z^{1/\alpha}\eta}\right)dz,\ \ (y_1,y_2)\neq(0,0). \quad (2.2)$$

It may be noted that Φ is the distribution function of standard normal random variable, Ω_α and P_α are the distribution and density functions of α-stable random variable with characteristic function $\exp(-\,|s|^{\alpha}\,)$.

The following theorem establishes the relation between $MLL_\alpha(\sigma,\eta)$ and bivariate operator stable distributions.

Theorem 2.1. *Let* $\{(Y_{1\,i},Y_{2\,i}),i\geq 1\}$ *be a sequence of* $MLL_\alpha(\sigma,\eta)$ *variables then the random vector* $(U_n,V_n)=((\frac{1}{n})^{1/2}\sum_{i=1}^{n}Y_{1i},(\frac{1}{n})^{1/\alpha}\sum_{i=1}^{n}Y_{2i})$ *is asymptotically distributed as bivariate symmetric stable with independent components.*

Proof: Let $\Pi_{(U_n,V_n)}(t,s)$ be the characteristic function of (U_n,V_n). Then by definition

$$\begin{aligned}\Pi_{(U_n,V_n)}(t,s) &= \left[\Pi_{(Y_{1\,i},Y_{2\,i})}(\,(\frac{1}{n})^{1/2}t,(\frac{1}{n})^{1/\alpha}s\,)\right]^n\\ &= \left[\frac{1}{1+\sigma^2\frac{t^2}{n}+\eta^\alpha\frac{|s|^\alpha}{n}}\right]^n.\end{aligned}$$

When $n\to\infty$, $\Pi_{(U_n,V_n)}(t,s)=\exp\left\{-(\sigma^2\,t^2+\eta^\alpha\,|s|^\alpha)\right\}$, which is the characteristic function of a bivariate symmetric stable distribution with independent components. Hence the theorem. □ Kozubowski *et al.* (2005) derived the property of stability of OGS distribution with respect to geometric summation and hence corresponding to $MLL_\alpha(\sigma,\eta)$ distribution we have the following theorem.

Theorem 2.2. *Let* $\{(Y_{1\,i},Y_{2\,i}),i\geq 1\}$ *be a sequence of i.i.d random vectors and let* $N_p,p\in(0,1)$ *be a geometric random variable with mean* $\frac{1}{p}$, *independent of* $(Y_{1\,i},Y_{2\,i})$*'s. Then the random vectors* $(\,p^{1/2}U_{N_p},p^{1/\alpha}V_{N_p})\underline{\underline{d}}\ (Y_{1i},Y_{2i})$ *for all* p, *if and only if* $(Y_{1\,i},Y_{2\,i})\underline{\underline{d}}\,MLL_\alpha(\sigma,\eta)$, *where* $U_{N_p}=\sum_{i=1}^{N_p}Y_{1\,i}$ *and* $V_{N_p}=\sum_{i=1}^{N_p}Y_{2\,i}$.

Proof: Proof follows easily using the arguments in Kozubowski *et al.* (2005).

Now we obtain a characterization of $MLL_\alpha(\sigma,\eta)$ random vector using bivariate compounding. Define $U_{N_1} = \sum_{i=1}^{N_1} Y_{1\,i}$ and $U_{N_2} = \sum_{i=1}^{N_2} Y_{2\,i}$, where (N_1, N_2) has bivariate geometric distribution with distribution function

$$P(N_1 > n_1, N_2 > n_2) = \begin{cases} p_{11}^{n_1}(p_{01}+p_{11})^{n_2-n_1} & if\ n_1 \leq n_2 \\ p_{11}^{n_2}(p_{10}+p_{11})^{n_1-n_2} & if\ n_1 \geq n_2, \end{cases} \tag{2.3}$$

where $0 < p_{01}, p_{10}, p_{11} < 1;\ \ p_{01}+p_{10}+p_{11} \leq 1;\ n_1, n_2 = 1, 2, \ldots$(see Block(1977)).

□

Theorem 2.3. *The distribution of* $(Y_{1\,i}, Y_{2\,i})$ *is* $MLL_\alpha(\sigma, \eta)$ *if and only if* ($p_{01}^{1/2} U_{N_1}$, $p_{10}^{1/\alpha} U_{N_2}$), *where* $U_{N_1} = \sum_{i=1}^{N_1} Y_{1\,i}$ *and* $U_{N_2} = \sum_{i=1}^{N_2} Y_{2\,i}$, *is distributed as marginal Laplace and Linnik distribution with independent marginals, where* (N_1, N_2) *has bivariate distribution (2.3) with* $p_{01}+p_{10}+p_{11} = 1$.

Proof: The characteristic function of (U_{N_1}, U_{N_2}) is given by

$$\Psi(t,s) = \Pi(t,s)\,[p_{01}\Psi(0,s) + p_{10}\Psi(t,0) + p_{11}\Psi(t,s)] \tag{2.4}$$

where $\Pi(t,s)$ is the characteristic function of $(Y_{1\,i}, Y_{2\,i})$. Hence

$$\Psi(t,0) = \frac{(1-p_{10}-p_{11})\Pi(t,0)}{1-(p_{10}+p_{11})\Pi(t,0)} \ \text{ and } \ \Psi(0,s) = \frac{(1-p_{01}-p_{11})\Pi(0,s)}{1-(p_{01}+p_{11})\Pi(0,s)} \tag{2.5}$$

Let $(Y_{1\,i}, Y_{2\,i}) \stackrel{d}{=} MLL_\alpha(\sigma, \eta)$, then

$$\Pi(t,s) = \frac{1}{1+\sigma^2 t^2 + \eta^\alpha\ |s|^\alpha}.$$

Substituting $\Pi(t,0)$ and $\Pi(0,s)$ in (2.5) we get,

$$\Psi(t,0) = \frac{p_{01}}{p_{01}+\sigma^2 t^2} \text{ and } \Psi(0,s) = \frac{p_{10}}{p_{10}+\eta^\alpha\ |s|^{\,\alpha}}.$$

Hence,

$$p_{01}^{1/2} U_{N_1} \stackrel{d}{=} Y_{1\,i} \text{ and } p_{10}^{1/\alpha} U_{N_2} \stackrel{d}{=} Y_{2\,i}.$$

To prove converse, substitute $\Psi(t,s)$, $\Psi(t,0)$ and $\Psi(0,s)$ in (2.4). This yields $\Pi(t,s)$ as $\Pi(t,s) = \dfrac{1}{1+\sigma^2 t^2 + \eta^\alpha\ |s|^\alpha}$. Hence the theorem. □

2.2 Bivariate Autoregressive MLL Process

In various applications it is frequently of interest to study time series models for bivariate data and so autoregressive models are extended to bivariate time series models by several researchers. Dewald *et al.* (1989) developed and studied a bivariate exponential autoregressive process, which is broader and simpler than the models described by Gaver and Lewis (1980), and Raftery (1982). Block *et al.* (1988) introduced additive first order autoregressive bivariate exponential and geometric processes and studied their properties. Ristic and Popovic (2003) discussed a bivariate autoregressive process with uniform marginal distribution and examined the autocorrelation structure of the process. In this section we develop a time series model that can be used for modeling heavy tailed bivariate time series observations with different tail behavior for each component.

We now define a bivariate AR (1) process $\{(X_n, Y_n)\}$as follows:

Let $\{(\varepsilon_n, \eta_n), n \geq 1\}$ be a sequence of independent and identically distributed bivariate random vectors and let (X_0, Y_0) be a $MLL_\alpha(\sigma, \eta)$ random vector with characteristic function (1.2). Define $\{(X_n, Y_n)\}$by the first order autoregressive structure

$$X_n = p^{1/2} X_{n-1} + \varepsilon_n \, and \, Y_n = p^{1/\alpha} Y_{n-1} + \eta_n, n = 1, 2, \ldots \tag{2.6}$$

where $0 \leq p < 1$.Note that (X_i, Y_i) is independent of (ε_n, η_n) for $i < n$.

Let $\Pi_{(X_n, Y_n)}(t, s)$ and $\Pi_{(\varepsilon_n, \eta_n)}(t, s)$ be the characteristic functions of (X_n, Y_n) and (ε_n, η_n) respectively. Then (2.6) gives

$$\Pi_{(\varepsilon_n, \eta_n)}(t, s) = \frac{\Pi_{(X_n, Y_n)}(t, s)}{\Pi_{(X_{n-1}, Y_{n-1})}(p^{1/2} t, p^{1/\alpha} s)} \quad . \tag{2.7}$$

Let $\{(X_n, Y_n)\}$ be stationary sequence with $MLL_\alpha(\sigma, \eta)$ marginal distribution. Then from (2.7)

$$\begin{aligned} \Pi_{(\varepsilon_n, \eta_n)}(t, s) &= \frac{1 + \sigma^2 \, p \, t^2 + \eta^\alpha \, p \, |s|^\alpha}{1 + \sigma^2 \, t^2 + \eta^\alpha \, |s|^\alpha} \\ &= p + (1 - p) \frac{1}{1 + \sigma^2 \, t^2 + \eta^\alpha \, |s|^\alpha}. \end{aligned}$$

Hence

$$(\varepsilon_n, \eta_n) = \begin{cases} (0, 0) & \text{w.p.} \quad p \\ MLL_\alpha(\sigma, \eta) & \text{w.p.} \; 1 - p. \end{cases} \tag{2.8}$$

Also it can be verified that, if $(X_0, Y_0) \underline{\underline{d}} MLL_\alpha(\sigma, \eta)$ and $\{(\varepsilon_n, \eta_n), n \geq 1\}$ is a sequence of independent and identically distributed bivariate random variables given by (2.8), the first order autoregressive process (2.6) is stationary with MLL marginal distribution.

Thus we have the following theorem.

Theorem 2.4. *Let $\{(\varepsilon_n, \eta_n), n \geq 1\}$ be a sequence of independent and identically distributed random vectors distributed as (2.8), and $(X_0, Y_0) \underline{\underline{d}} MLL_\alpha(\sigma, \eta)$. Then (2.6) defines a stationary bivariate time series with $MLL_\alpha(\sigma, \eta)$ marginal distribution.*

We call the process (2.6) as first order MLL autoregressive process (MLLARP (1)). The equation (2.6) can be represented in the vector form

$$Z_n = M\ Z_{n-1} + \ W_n, \tag{2.9}$$

where

$$Z_n = (X_n, Y_n)', \quad M = \begin{bmatrix} p^{1/2} & 0 \\ 0 & p^{1/\alpha} \end{bmatrix}, \ \ \textit{and} \ \ W_n = (\varepsilon_n, \eta_n)'.$$

It can be noted that the bivariate process can be represented as

$$X_n = (p^{1/2})^n X_0 + \sum_{k=0}^{n-1} (p^{1/2})^n \varepsilon_{n-k} \ \text{ and } \ Y_n = (p^{1/\alpha})^n Y_0 + \sum_{k=0}^{n-1} (p^{1/\alpha})^n \eta_{n-k}.$$

Hence, in terms of characteristic function we can write

$$\Pi_{(X_n,Y_n)}(t,s) = \Pi_{(X_0,Y_0)}(\,(p^{1/2})^n t, (p^{1/\alpha})^n s\,) \prod_{k=0}^{n-1} \Pi_{(\varepsilon_{k+1},\eta_{k+1})}(\,(p^{1/2})^k t, (p^{1/\alpha})^k s\,).$$

From this expression it is clear that if (X_0, Y_0) is distributed arbitrary and $\{(\varepsilon_n, \eta_n), n \geq 1\}$ is a sequence of independent and identically distributed bivariate random variables given by (2.8), then the first order autoregressive process (2.6) is asymptotically stationary with MLL marginal distribution.

We can note that the characteristic function (1.2) when $\alpha = 2$ is

$$\Psi(t,s) = \frac{1}{1 + \sigma^2\, t^2 + \eta^2\, s^2},$$

which is the characteristic function of bivariate symmetric Laplace (BSL) distribution with mean vector $\mathbf{0}$ and variance-covariance matrix $\sum = \begin{pmatrix} 2\sigma^2 & 0 \\ 0 & 2\eta^2 \end{pmatrix}$ (see, Kotz et al. (2001). In this case the bivariate autoregressive process defined by (2.6) with BSL marginal distribution have the solution

$$(\varepsilon_n, \eta_n) = \begin{cases} (0,0) & \text{w.p.} \quad p \\ (\delta_n, \gamma_n) & \text{w.p.} \quad 1-p \end{cases}$$

where (δ_n, γ_n) is BSL random vector with characteristic function

$$\Psi(t,s) = \frac{1}{1+\sigma^2\, t^2 + \eta^2\, s^2}.$$

In this case, we can study the second order properties of bivariate process using the vector representation (2.9). The auto covariance matrix $\Gamma(h)$ is given by

$$\begin{aligned} \Gamma(h) &= Cov(Z_n, Z_{n-h}) = \begin{pmatrix} Cov(X_n, X_{n-h}) & Cov(X_n, Y_{n-h}) \\ Cov(Y_n, X_{n-h}) & Cov(Y_n, Y_{n-h}) \end{pmatrix} \\ &= M\Gamma(h-1), \text{using } (2.9) \\ &= M^h\, \Gamma(0), \text{ where } \Gamma(0) = \begin{pmatrix} 2\sigma^2 & 0 \\ 0 & 2\eta^2 \end{pmatrix}. \end{aligned}$$

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3 Bivariate Marginal asymmetric Laplace and asymmetric Linnik Processes

In practical situation we may come across the situation where empirical distribution of the components of bivariate vector appears to be asymmetric, with steep peak and tails heavier than those allowed by normal distribution. In such situation a skewed form of bivariate distribution is well suited for modeling the data. Here we consider a class of bivariate distributions, which is a member of class of marginally geometric stable laws, with asymmetric Laplace and asymmetric Linnik distributions as marginals. If the characteristic function of a bivariate random vector $Y = (Y_1, Y_2)$ has characteristic function

$$\Psi(t,s) = \frac{1}{1+\sigma^2 t^2 + \eta^\alpha |s|^\alpha - i\mu t - i\nu s}, \text{ where } \sigma, \eta \geq 0, \mu, \nu \in \mathrm{R}, \alpha \in (0,2], \quad (3.1)$$

then we say that Y follows marginal asymmetric Laplace and asymmetric Linnik distribution, and denote it by $Y \underline{\underline{d}}\ MALAL_\alpha(\mu, \nu, \sigma, \eta)$.

Remark 3.1. *When $\alpha = 2$, the characteristic function (3.1) reduced to a characteristic function of bivariate asymmetric Laplace distribution $BAL(\mu, \nu, \sigma, \eta, \rho)$ with $\rho = 0$ (see, Kotz et al. (2001)).*

Now we define a first order bivariate autoregressive process with $MALAL_\alpha$ (μ, ν, σ, η) marginal distribution. If we define the process with structure (2.6), then it seems to be difficult to get the solution of the bivariate innovation sequence. Hence we are introducing a bivariate autoregressive process with structural relationship equivalent to the one-parameter TEAR (1) model discussed in Lawrance and Lewis (1981). This construction is based on the property of geometric infinite divisibility satisfied by the OGS laws.

Let $\{(\varepsilon_n, \eta_n), n \geq 1\}$be a sequence of independent and identically distributed bivariate random vectors and let (X_0, Y_0) be a $MALAL_\alpha(\mu, \nu, \sigma, \eta)$ random vector with characteristic function (3.1). Define $\{(X_n, Y_n), n \geq 1\}$ as

$$X_n = \begin{cases} \varepsilon_n & \text{w.p.} \quad p \\ X_{n-1} + \varepsilon_n & \text{w.p.} \ 1-p \end{cases} \quad \text{and } Y_n = \begin{cases} \eta_n & \text{w.p.} \quad p \\ Y_{n-1} + \eta_n & \text{w.p.} \quad 1-p, \end{cases} \quad (3.2)$$

where$0 < p < 1$.

Let $\Pi_{(X_n, Y_n)}(t,s)$and $\Pi_{(\varepsilon_n, \eta_n)}(t,s)$be the characteristic function of (X_n, Y_n) and (ε_n, η_n) respectively. Then (3.2) gives

$$\Pi_{(\varepsilon_n,\eta_n)}(t,s) = \frac{\Pi_{(X_n,Y_n)}(t,s)}{p + (1-p)\Pi_{(X_{n-1},Y_{n-1})}(t,s)} \quad . \quad (3.3)$$

If $\{(X_n, Y_n)\}$be stationary sequence with $MALAL_\alpha(\mu, \nu, \sigma, \eta)$ marginal distribution, then from (3.3) we get

$$\Pi_{(\varepsilon_n,\eta_n)}(t,s) = \frac{1}{1+\sigma^2 p\ t^2 + \eta^\alpha p\ |s|^\alpha - i\mu p t - i\nu p s}.$$

Hence,

$$(\varepsilon_n, \eta_n) \underline{\underline{d}} \ MALAL_\alpha(\mu p, \nu p, \sigma p^{1/2}, \eta p^{1/\alpha}). \tag{3.4}$$

Also it can be verified that, if $(X_0, Y_0) \underline{\underline{d}} MALAL_\alpha(\mu, \nu, \sigma, \eta)$ and $\{(\varepsilon_n, \eta_n), n \geq 1\}$ is a sequence of independent and identically distributed bivariate random variables given by (3.4)), the first order autoregressive process (3.2) is stationary with $MALAL_\alpha(\mu, \nu, \sigma, \eta)$ marginal distribution.

Hence we have the following theorem.

Theorem 3.1. *Let $\{(\varepsilon_n, \eta_n), n \geq 1\}$ be a sequence of independent and identically distributed random vectors given by (3.4), and $(X_0, Y_0) \ \underline{\underline{d}} \ MALAL_\alpha(\mu, \nu, \sigma, \eta)$. Then the relation (3.2) defines a stationary bivariate time series with marginal distribution $MALAL_\alpha(\mu, \nu, \sigma, \eta)$.*

We call the process (3.2) as first order $MALAL$ autoregressive process ($MALAL$ARP (1)). When $\alpha = 2$, the relation (3.2) defines a first order autoregressive process with bivariate asymmetric Laplace distribution as marginal distribution.

4 Generalized marginal Laplace and Linnik process

Let $\{Y_i\}$ be a sequence of independent and identically distributed bivariate $MLL_\alpha(\sigma, \eta)$ random vectors with characteristic function (1.2). Then their sum $S_n = \sum\limits_{i=1}^{n} Y_i$ has the characteristic function $\Psi_{S_n}(t, s) = \left(\dfrac{1}{1 + \sigma^2 t^2 + \eta^\alpha \ |s|^\alpha}\right)^n$. By infinite divisibility character of OGS laws, the function $\Psi_{S_n}(t, s)$ is a legitimate characteristic function even when n is not an integer, but positive. Hence the function

$$\Psi(t, s) = \left(\frac{1}{1 + \sigma^2 t^2 + \eta^\alpha \ |s|^\alpha}\right)^\tau, (t, s) \in \mathrm{R}^2 \quad , \tag{4.1}$$

is a characteristic function for any $\sigma, \eta \geq 0, \alpha \in (0, 2], \tau \geq 0$. The corresponding bivariate random vector $Y = (Y_1, Y_2)$ is said to follow generalized marginal Laplace and Linnik distribution and represent it as $Y \underline{\underline{d}} GMLL_\alpha(\sigma, \eta, \tau)$. This class of distributions is a special case of operator υ-stable laws discussed in Kozubowski *et al.* (2003), when υ is a gamma distribution. Using the result in Kozubowski *et* *al.* (2003), $GMLL_\alpha(\sigma,\eta,\tau)$ random vector $Y=(Y_1,Y_2)$ admits the representation $Y \underline{\underline{d}}\,(W^{1/2}X_1, W^{1/\alpha}X_2)$ where the random variables W has gamma distribution with probability density function $\dfrac{w^{\tau-1}e^{-w}}{\Gamma(\tau)}$, X_1 and X_2 have normal and symmetric stable distributions with respective characteristic functions $\exp(-\sigma^2\, t^2)$ and $\exp(-\eta^\alpha\ |s|^\alpha)$. It may be noted that $W, X_1$and X_2 are mutually independent. From the above representation we can derive the distribution and density functions of $GMLL_\alpha(\sigma,\eta,\tau)$ random vector and given by

$$F(y_1,y_2)=\frac{1}{\Gamma(\tau)}\int_0^\infty \Phi(\frac{y_1}{\sqrt{2\,w}\,\sigma})\Omega_\alpha(\frac{y_2}{w^{1/\alpha}\eta})\,w^{\tau-1}e^{-w}dw,\ \ (y_1,y_2)\in \mathrm{R}^2 \tag{4.2}$$

$$f(y_1,y_2)=\frac{1}{2\sigma\,\eta\,\sqrt{\pi}\,\Gamma(\tau)}\int_0^\infty w^{(\tau-\frac{1}{\alpha}-\frac{1}{2})-1}\,e^{-w-\frac{y_1^2}{4\sigma^2 w}}P_\alpha(\frac{y_2}{w^{1/\alpha}\eta})dw, (y_1,y_2)\neq(0,0) \tag{4.3}$$

where Φ, Ω_α and P_α are as in Section 2.

Remark 4.1. *The generalized marginal Laplace and Linnik distribution can be generalized in a more natural way by introducing asymmetric component in the marginal distributions. The resulting distribution of random vector, namely generalized marginal asymmetric Laplace and asymmetric Linnik distribution, (in short $GMALAL$ distribution) have the characteristic function*

$$\Psi(t,s)=\left(\frac{1}{1+\sigma^2\,t^2+\eta^\alpha\ |s|^\alpha-i\mu t-i\nu s}\right)^\tau, (t,s)\in\mathrm{R}^2. \tag{4.4}$$

In this case the bivariate $GMALAL$ random vector $Y=(Y_1,Y_2)$ is represented as $Y\ \underline{\underline{d}}\ (W^{1/2}X_1+\mu\,W,\ W^{1/\alpha}X_2+\nu\,W)$, where the random variables W, X_1 and X_2 are as specified above.

Now we present a bivariate stationary first order autoregressive model with $GMLL_\alpha(\sigma,\eta,\tau)$ marginal distribution, given by the structure (2.6). Since $\{(X_n,Y_n),\ n\geq 1\}$ is distributed as $GMLL_\alpha(\sigma,\eta,\tau)$ distribution, from (4.1) the characteristic function of (ε_n,η_n) is given by

$$\begin{aligned}\Pi_{(\varepsilon_n,\eta_n)}(t,s) &= \left(\frac{1+\sigma^2\,p\,t^2+\eta^\alpha\,p\ |s|^\alpha}{1+\sigma^2\ t^2+\eta^\alpha\ \ |s|^\alpha}\right)^\tau\\ &= \left(p+(1-p)\frac{1}{1+\sigma^2\ t^2+\eta^\alpha\ \ |s|^\alpha}\right)^\tau.\end{aligned} \tag{4.5}$$

This implies that $\{(\varepsilon_n, \eta_n), n \geq 1\}$ is distributed independently and identically as τ- fold convolution of random vectors (δ_n, γ_n), where

$$(\delta_n, \gamma_n) = \begin{cases} (0,0) & \text{w.p.} \quad p \\ MLL_\alpha(\sigma, \eta) & \text{w.p.} \quad 1-p \end{cases}. \tag{4.6}$$

Also it can be verified that, if $(X_0, Y_0) \underline{\underline{d}} GMLL_\alpha(\sigma, \eta, \tau)$ and $\{(\varepsilon_n, \eta_n), n \geq 1\}$ is a sequence of independent and identically distributed τ-fold convolution of (δ_n, γ_n), where (δ_n, γ_n) is defined by (4.6), the first order bivariate autoregressive process (2.6) is stationary with $GMLL_\alpha(\sigma, \eta, \tau)$ marginal distribution.

Thus we have the following theorem

Theorem 4.1. *Let $\{(\varepsilon_n, \eta_n), n \geq 1\}$is distributed independently and identically as τ-fold convolution of random vectors (δ_n, γ_n), where $(\delta_n, \gamma_n)$is defined by (4.6), with $(X_0, Y_0) \underline{\underline{d}} GMLL_\alpha(\sigma, \eta, \tau)$. Then the relation (2.6) defines a stationary bivariate time series with $GMLL_\alpha(\sigma, \eta, \tau)$ marginal distribution.*
Note that when τ=1, the first order $GMLL$ autoregressive process is reduced to $MLLARP$ (1) defined in section 2.

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On Measure of Rotatability for Second Order Response Surface Designs – A Review

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ABSTRACT

Measure of rotatability that enables us to know the degree of rotatability for a given response surface design. In this paper, a review on measure of rotatability for second order response surface designs is studied. Further, different methods of constructions of measure of rotatability of second order response surface designs like measure of rotatability using central composite designs, balanced incomplete block designs, pairwise balanced designs, symmetrical unequal block arrangements with two unequal block sizes and pair of partially balanced incomplete block designs are examined in detail.

Key words and Phrases: Second order response surface designs, second order, rotatable designs, measure of rotatability.

1 Introduction

Response surface methodology is a statistical technique that is very useful in design and analysis of scientific experiments. In many experimental situations the experimenter is concerned with explaining certain aspects of a functional relationship $Y = f(x_1, x_2, \ldots, x_v) + e$, where Y is the response and $x_1, x_2, \ldots, x_v$ are the

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levels of v-quantitative variables or factors and e is the random error. Response surface methods are useful where several independent variables influence a dependent variable. The independent variables are assumed to be continuous and controlled by the experimenter. The response is assumed to be as random variable. For example, if a chemical engineer wishes to find the temperature (x_1) and pressure (x_2) that maximizes the yield (response) of his process, the observed response Y may be written as a function of the levels of the temperature (x_1) and pressure (x_2) as $Y = f(x_1, x_2) + e$.

Box and Hunter (1957) introduced multifactor experimental designs for exploring response surface designs. Das and Narasimham (1962) constructed rotatable designs through balanced incomplete block designs (BIBD). Raghavarao (1963), Tyagi (1964), Chowdhury and Gupta (1985), Victorbabu (2004) and several others have suggested various methods for the construction of SORD. Draper and Guttman (1988) suggested an index of rotatability. Khuri (1988) suggested a measure of rotatability for response surface designs. Draper and Pukelsheim (1990) suggested another look at rotatability. Specifically, Park et al. (1993) introduced a new measure of rotatability for second order response surface designs and illustrated for 3k factorial and central composite designs. Victorbabu and Surekha (2013, 2014) suggested a measure of rotatability for second order response surface designs using BIBD, incomplete block designs like pairwise balanced designs (PBD), symmetrical unequal block arrangements (SUBA) with two unequal block sizes and a pair of partially balanced incomplete block designs (PBIBD). In this paper, a review on measure of rotatability for second order response surface designs is studied. Different methods of constructions of measure of rotatability of second order response surface designs like measure of rotatability using central composite designs (Park et al. (1993)), balanced incomplete block designs (Victorbabu and Surekha(2015)), pairwise balanced designs (Victorbabu and Surekha (2013)), symmetrical unequal block arrangements with two unequal block sizes (Victorbabu and Surekha (2013)) and pair of partially balanced incomplete block designs (Victorbabu and Surekha (2014)) are examined in detail.

2 Second order rotatable designs

Suppose we want to use the second order response surface design D=x_{iu} to fit the surface,

$$Y_u \;=\; b_0 + \sum_{i=1}^{v} b_i\, x_{iu} + \sum_{i=1}^{v} b_{ii}\, x_{iu}^2 + \sum\sum_{i<j} b_{ij}\, x_{iu}x_{ju} + \, e_u, \tag{2.1}$$

where x_{iu} denotes the level of the i^{th} factor (i =1,2,. . . ,v) in the u^{th} run (u=1,2,. . . ,N) of the experiment, $e_u^{\prime}s$ are uncorrelated random errors with mean zero and variance σ^2, is said to be second order rotatable design (SORD) if the variance of the estimate of $Y_u(x_1, x_2, \ldots, x_v)$ with respect to each of independent variables (x_i) is only a function of the distance $(d^2 = \sum_{i=1}^{v} x_i^2)$ of the point $(x_1, x_2, \ldots, x_v)$ from the origin (center) of the design. Such a spherical variance function for estimation of responses in the second order response surface is achieved if the design points satisfy the following conditions (Box and Hunter (1957), Das and Narasimham (1962)).

$$\begin{aligned}
\sum_{u=1}^{N}\prod_{i=1}^{v} x_{iu}^{\alpha_i} &= 0 \,\text{if any}\, \alpha_i \,\text{is odd, for} \sum \alpha_i \le 4, && (2.2)\\
\sum x_{iu}^2 &= \text{constant} = N\lambda_2, && (2.3)\\
\sum x_{iu}^4 &= \text{constant} = cN\lambda_4; \text{for all i}, && (2.4)\\
\sum x_{iu}^2 x_{ju}^2 &= \ \text{constant} = N\lambda_4; \text{for} i \ne j, && (2.5)\\
(c+v-1)\lambda_4 &> v\lambda_2^2, && (2.6)\\
\sum x_{iu}^4 &= c\sum x_{iu}^2 x_{ju}^2, && (2.7)
\end{aligned}$$

where c, λ_2 and λ_4 are constants.

The variances and covariances of the estimated parameters are

$$\begin{aligned}
V(\hat{b_0}) &= \frac{\lambda_4(c+v-1)\sigma^2}{N\left[\lambda_4(c+v-1)-v\lambda_2^2\right]},\ V(\hat{b_i}) = \frac{\sigma^2}{N\lambda_2},\ V(\hat{b_{ij}}) = \frac{\sigma^2}{N\lambda_4}, \\
V(\hat{b_{ii}}) &= \frac{\sigma^2}{(c-1)N\lambda_4}\left[\frac{\lambda_4(c+v-2)-(v-1)\lambda_2^2}{\lambda_4(c+v-1)-v\lambda_2^2}\right], \\
Cov(\hat{b_0},\hat{b_{ii}}) &= \frac{-\lambda_2\sigma^2}{N\left[\lambda_4(c+v-1)-v\lambda_2^2\right]}, \\
Cov(\hat{b_{ii}},\hat{b_{jj}}) &= \frac{\lambda_2^2-\lambda_4^2\sigma^2}{(c-1)N\lambda_4\left[\lambda_4(c+v-1)-v\lambda_2^2\right]} \qquad (2.8)
\end{aligned}$$

and other covariances vanish.

The variances of the estimated response at the point $(x_{10}, x_{20}, \ldots, x_{v0})$ is

$$\begin{aligned}
V(\hat{y_0}) &= V(\hat{b_0}) + \left[V(\hat{b_i}) + 2cov(\hat{b_0}\hat{b_{ii}})\right]d^2 + V(\hat{b_{ii}})d^4 \\
&\quad + \sum x_{i0}^2 x_{j0}^2\left[V(\hat{b_{ij}}) + 2cov(\hat{b_{ii}}\hat{b_{jj}}) - 2V(\hat{b_{ii}})\right] \qquad (2.9)
\end{aligned}$$

The coefficient of $\sum x_{i0}^2 x_{j0}^2$ in the above equation (2.9) is simplified to $\frac{(c-3)\sigma^2}{(c-1)N\lambda_4}$.

A second order response surface design D is said to be a SORD, if in this design $c = 3$ and all the conditions (2.2) to (2.8) hold.

3 Measure of rotatability for second order response surface designs

Following Box and Hunter (1957), Das and Narasimham (1962), Park et al. (1993), equations from (2.2) to (2.8) give the necessary and sufficient conditions for measure of rotatability for any general second order response surface designs. Further we have, $V(\hat{b_i})$ are equal for all i,

$V(\hat{b_{ii}})$ are equal for all i,

$V(\hat{b_{ij}})$ are equal for i, j, where $i \neq j$,

$$Cov(\hat{b_i},\hat{b_{ii}}) = Cov(\hat{b_i},\hat{b_{ij}}) = Cov(\hat{b_{ii}},\hat{b_{ij}}) = Cov(\hat{b_{ij}},\hat{b_{il}}) = 0 \text{ for all } i \neq j \neq l. \qquad (3.1)$$

Park, et. al. (1993) suggested that if the conditions in (2.2) to (2.8) and (3.1) are met, then the following measure ($P_v(D)$) given below asses the degree of measure of rotatability for any general second order response surface design (see, Park et al., 1993, p. 661).

$$P_v(D) = \frac{1}{R_v(D)}, \tag{3.2}$$

where

$$R_v(D) = \left[\frac{N}{\sigma^2}\right]^2 \frac{6v\left[V(\hat{b_{ij}}) + 2cov(\hat{b_{ii}}\hat{b_{jj}}) - 2V(\hat{b_{ii}})\right]^2 (v-1)}{(v+2)^2(v+4)(v+6)(v+8)g^8}. \tag{3.3}$$

On simplification of numerator of (3.3), $\left[V(\hat{b_{ij}}) + 2cov(\hat{b_{ii}}\hat{b_{jj}}) - 2V(\hat{b_{ii}})\right]$ becomes $\frac{(c-3)\sigma^2}{(c-1)N\lambda_4}$. Thus, $R_v(D)$ becomes

$$R_v(D) = \left[\frac{c-3}{c-1}\right]^2 \frac{6v(v-1)}{{\lambda_4}^2(v+2)^2(v+4)(v+6)(v+8)g^8},$$

where g is the scaling factor.

4 Measure of rotatability for second order response surface designs using central composite designs

Central composite designs are constructed by adding suitable factorial combinations to those obtained from $\frac{1}{2^p} \times 2^v$ fractional factorial design (here $\frac{1}{2^p} \times 2^v$ denotes a suitable fractional replicate of 2^v), in which no interaction with less than five factors is confounded. In coded form the points of $2^v(2^{t(v)})$ factorial have coordinates (±1, ±1, ..., ±1) and 2v axial points have coordinates of the form (±a, 0,..., 0), (0, ±a,..., 0), ..., (0, 0, ..., ±a) etc., and if necessary n_0 central points may be replicated sometimes. Measure of rotatability for second order response surface designs using central composite designs is obtained by the following result (see, Park et al., 1993, p. 661)

Result 1: The design points $(\pm 1, \pm 1, ..., \pm 1)2^{\mathrm{t(v)}}\mathrm{U}(\pm \mathrm{a}, 0, \ldots, 0)2^1\mathrm{Un_0}$ will give a v-dimensional measure of rotatability for second order response surface designs using central composite design in $N = 2^{t(v)} + 2v + n_0$ design points with level 'a' pre-fixed,

$$c = \frac{2^{t(v)}+2a^4}{2^{t(v)}} \text{ and } P_v(D) = P_v(D) = \frac{1}{R_v(D)},$$

where

$$R_v(D) = \left[\frac{c-3}{c-1}\right]^2 \frac{6v(v-1)}{\lambda_4{}^2(v+2)^2(v+4)(v+6)(v+8)g^8}.$$

$$\text{The Scaling factor } g = \begin{cases} \frac{1}{a}, & if\, a \geq \sqrt{v} \\ \frac{1}{\sqrt{v}}, & if a < \sqrt{v} \end{cases}$$

Thus the measure of rotatability for second order response surface designs is one if and only if a design is rotatable (i.e., $c = 3$). Table 1 gives the values of measure of rotatability for second order response surface designs using central composite designs.

5 Measure of rotatability for second order response surface designs using BIBD

Let (v, b, r, k, λ) denotes a BIBD, $2^{t(k)}$ denotes a fractional replicate of 2^k in ± 1 levels, in which no interaction with less than five factors is confounded. $[1-(\mathrm{v,b,r,k},\lambda)]$ denote the design points generated from the transpose of incidence matrix of BIBD. $[1-(\mathrm{v,b,r,k},\lambda)]\ 2^{t(k)}$ are the $b^{2t(k)}$ design points generated from BIBD by "multiplication" (see, Raghavarao, (1971), pp. 298-300). $(\pm a, 0, 0, \ldots, 0)$ 21 denote the design points generated from $(\pm a, 0, 0, \ldots, 0)$ point set, and $\bigcup$ denotes combination of the design points generated from different sets of points. n_0 denotes the number of central points. Measure of rotatability for second order response surface designs using BIBD is obtained by the following result (see, Victorbabu and Surekha, 2015).

Result 2: The design points,$[\,1-(\,\mathrm{v,b,r,k},\lambda)\,]\ \ 2^{t(k)} \bigcup (\pm a, 0, \ldots, 0)\, 2^1 \bigcup_0$ will give a v-dimensional measure of rotatability for second order response surface designs using BIBD in $N = b2^{t(k)} + 2v + n_0$ design points, with level 'a' pre-fixed, $c = \frac{r2^{t(k)}+2a^4}{\lambda 2^{t(k)}}$ and

$$P_v(D) = \frac{1}{R_v(D)},$$

where

$$R_v(D) = \left[\frac{c-3}{c-1}\right]^2 \frac{6v(v-1)}{{\lambda_4}^2(v+2)^2(v+4)(v+6)(v+8)g^8}.$$

$$\text{The scaling factor g} = \begin{cases} \frac{1}{a},\ if\ a < \sqrt{2^{t(k)-1}(b-r)+v} \\ \frac{1}{\sqrt{2^{t(k)-1}(b-r)+v}},\ if a \geq \sqrt{2^{t(k)-1}(b-r)+v} \end{cases}.$$

Table 2 gives the values of measure of rotatability for second order response surface designs using BIBDs.

6 Measure of rotatability for second order response surface designs using PBD

Let $(\mathrm{v, b, r, k_1, k_2, \ldots, k_p}, \lambda)$ denote a PBD, $k = \sup(k_1, k_2, \ldots, k_p)$ and $2^{\mathrm{t(k)}}$ denote a fractional replicate of 2^k in ± 1 levels, in which no interaction with less than five factors is confounded. $[1 - (\mathrm{v, b, r, k_1, k_2, \ldots, k_p}, \lambda)]$ denote the design points generated from the transpose of incidence matrix of PBD. $[1 - (\mathrm{v, b, r, k_1, k_2, \ldots, k_p}, \lambda)]\ 2^{\mathrm{t(k)}}$ are the $\mathrm{b}2^{\mathrm{t(k)}}$ design points generated from PBD by "multiplication", $(\pm\mathrm{a},0,0,\ldots,0)$ 2^1 denote the design points generated from $(\pm\mathrm{a}, 0, ..., 0)$ point set. Measure of rotatability for second order response surface designs using PBD is obtained by the following result (see, Victorbabu and Surekha, 2013).

Result 3: The design points, $[1 - (\mathrm{v, b, r, k_1, k_2, \ldots, k_p}, \lambda)]\ 2^{\mathrm{t(k)}} \bigcup (\pm\mathrm{a}, 0, ..., 0)\ 2^1$ $\bigcup(\mathrm{n_0})$ give a v-dimensional measure of rotatability for second order response surface designs using PBD in $\mathrm{N = b2^{t(k)} + 2v + n_0}$ design points, with level 'a' pre-fixed, $\mathrm{c} = \frac{\mathrm{r2^{t(k)} + 2a^4}}{\lambda 2^{\mathrm{t(k)}}}$ and

$$P_v(D) = \frac{1}{R_v(D)},$$

where

$$R_v(D) = \left[\frac{c-3}{c-1}\right]^2 \frac{6v(v-1)}{{\lambda_4}^2(v+2)^2(v+4)(v+6)(v+8)g^8}$$

and

$$\mathrm{g} = \begin{cases} \frac{1}{a},\ if\ a < \sqrt{2^{t(k)-1}(b-r)+v} \\ \frac{1}{\sqrt{2^{t(k)-1}(b-r)+v}},\ if a \geq \sqrt{2^{t(k)-1}(b-r)+v} \end{cases}.$$

Table 3 gives the values of measure of rotatability for second order response surface designs using PBDs.

7 Measure of rotatability for second order response surface designs using SUBA with two unequal block sizes

Let $(v, b, r, k_1, k_2, b_1, b_2, \lambda)$ denote a SUBA with two unequal block sizes, $b_1 + b_2$=b, $k = \sup(k_1, k_2)$ and $2^{t(k)}$ denote a fractional replicate of 2^kin ± 1 levels, in which no interaction with less than five factors is confounded. $[1 - (v, b, r, k_1, k_2, b_1, b_2, \lambda)]$ denote the design points generated from the transpose of incidence matrix of SUBA with two unequal block sizes. $[1 - (v, b, r, k_1, k_2, b_1, b_2, \lambda)]\ 2^{t(k)}$are the $b2^{t(k)}$ design points generated from SUBA with two unequal block sizes by "multiplication", $(\pm a,\ 0, 0, ..., 0)2^1$ denote the design points generated from $(\pm a,\ 0, 0, ..., 0)$ point set. Measure of rotatability for second order response surface designs using SUBA with two unequal block sizes is obtained by the following result (See, Victorbabu and Surekha, 2013).

Result 4: The design points, $[1 - (v, b, r, k_1, k_2, b_1, b_2, \lambda)]\ 2^{t(k)} \bigcup (a, 0, ..., 0)\ 2^1 \bigcup (n_0)$ give a v-dimensional measure of rotatability for second order response surface designs using SUBA with two unequal block sizes in $N = b2^{t(k)} + 2v + n_0$ design points, with level 'a' pre-fixed, $c = \frac{r2^{t(k)} + 2a^4}{\lambda 2^{t(k)}}$ and

$$P_v(D) = \frac{1}{R_v(D)},$$

where

$$R_v(D) = \left[\frac{c-3}{c-1}\right]^2 \frac{6v(v-1)}{{\lambda_4}^2(v+2)^2(v+4)(v+6)(v+8)g^8}$$

and

$$g = \begin{cases} \frac{1}{a},\ if\ a < \sqrt{2^{t(k)-1}(b-r)+v} \\ \frac{1}{\sqrt{2^{t(k)-1}(b-r)+v}},\ if a \geq \sqrt{2^{t(k)-1}(b-r)+v} \end{cases}.$$

Table 4 gives the values of measure of rotatability for second order response surface designs using SUBA with two unequal block sizes.

8 Measure of rotatability of second order response surface designs using a pair of PBIBD

Take an incomplete block arrangement with constant block size and replication in which some pair of treatments occur λ_{11} times each ($\lambda_{11} \neq 0$) and some other pairs do not occur at all ($\lambda_{12} = 0$) (the design need not be PBIBD). Take this as the first design. For the second design take the incomplete block design with all missing pairs (in the first design) once each with $k = 2$, $\lambda_{21} = 0$, $\lambda_{22} = 1$. Such pairs of designs can be construct in a straight forward manner using existing two-associate PBIB designs with one of the $\lambda's$ equal to zero.

Let $D_1 = (v, b_1, r_1, k_1, \lambda_{11} \neq 0, \lambda_{12} = 0)$ be an incomplete block design with constant replication in which only some pair of treatments occur a constant number of times $\lambda_{11}(\lambda_{12} = 0)$, $2^{t(k_1)}$ denote a fractional replicate of 2^{k_1} in +1 and -1 levels, in which no interaction with less than five factors is confounded. Let $[(1 - (v, b_1, r_1, k_1, \lambda_{11}, \lambda_{12} = 0)]$ denote the design points generated from the transpose of the incidence matrix of incomplete block design D_1. $[(1 - (v, b_1, r_1, k_1, \lambda_{11}, \lambda_{12} = 0)]2^{t(k_1)}$are the $b_1 2^{t(k_1)}$design points generate from D_1 by "multiplication". Let $D_2 = (v, b_2, r_2, k_2 = 2, \lambda_{21} = 0, \lambda_{22} = 1)$be the associated second design containing only the missing pairs of treatments of above design D_1. $[a - (v, b_2, r_2, k_2 = 2, \lambda_{21} = 0, \lambda_{22} = 1)2^2$ are the $b_2 2^2$ design points generated from D_2 by "multiplication". Measure of rotatability for second order response surface designs using a pair of PBIBD is obtained by the following result (see, Victorbabu and Surekha, 2014)

Result 5: The design points,

$$[\,1 - (v, b_1, r_1, k_1, \lambda_{11}, \lambda_{12}\,]\ 2^{\mathrm{t(k_1)}} \bigcup [\,a - (v, b_2, r_2, k_2 = 2, \lambda_{21}, \lambda_{22})\,]\ 2^2$$

give a v-dimensional measure of rotatability for second order response surface designs using a pair of PBIBD in $\mathrm{N} = \mathrm{b}_1 2^{\mathrm{t}(\mathrm{k}_1)} + \mathrm{b}_2 2^2$ design points, with level 'a' pre-fixed,

$$\mathrm{c} = \frac{\mathrm{r}_1 2^{\mathrm{t}(\mathrm{k}_1)} + \mathrm{r}_2 2^2 \mathrm{a}^4}{\lambda_{11} 2^{\mathrm{t}(\mathrm{k}_1)} + \lambda_{21} 2^2 \mathrm{a}^4}$$

and

$$P_v(D) = \frac{1}{R_v(D)},$$

where

$$R_v(D) = \left[\frac{c-3}{c-1}\right]^2 \frac{6v(v-1)}{{\lambda_4}^2 (v+2)^2 (v+4)(v+6)(v+8) g^8}$$

and

$$\mathrm{g} = \begin{cases} \frac{1}{\mathrm{a}} \text{, if a} < \sqrt{\frac{(\mathrm{b})1-\mathrm{r})1)2^{\mathrm{t}(\mathrm{k})1)-2}}{\mathrm{r})2} + \frac{\mathrm{b})2}{\mathrm{r})2}} \, ; \\ \frac{1}{\sqrt{\frac{(\mathrm{b})1-\mathrm{r})1)2^{\mathrm{t}(\mathrm{k})1)-2}}{\mathrm{r})2} + \frac{\mathrm{b})2}{\mathrm{r})2}}} \, , \quad \text{if a} \geq \sqrt{\frac{(\mathrm{b})1-\mathrm{r})1)2^{\mathrm{t}(\mathrm{k})1)-2}}{\mathrm{r})2} + \frac{\mathrm{b})2}{\mathrm{r})2}} \end{cases} .$$

Table 5 gives the values of measure of rotatability for second order response surface designs using a pair of PBIBDs. It may be pointed out here that measure of rotatability for second order response surface designs using a pair of PBIBD sometimes leads to designs with less number of design points.

9 Concluding Remarks

A measure of rotatability that enables us to know the degree of rotatability for a given second order response surface design is studied. The measure of rotatability is one if and only if a design is rotatable, and it is smaller than one for a non-rotatable design. Different methods of constructions of measure of rotatability for second order response surface designs were examined in detail. These methods are useful in selecting a proper design for second order response surface polynomial models for the construction of measure of rotatability for second order response surface designs with minimum number of design points.

There is scope for further research to evolve new methods of constructions of measure of rotatability for second order response surface designs which will lead to designs with lesser number of design points for different 'v' compared as per the existing methods of construction vide tables.

Different methods of constructions of measure of rotatability for second order response surface designs for different values of v are given in Table 1 to 5.

Table 1: Measure of rotatability for second order response surface designs using central composite designs (See, Park et al. 1993; Victorbabu and Surekha, 2012)

v=2, N=9, a*=1.41421356				
a	c	g	Rv(D)	Pv(D)
1	1.5	0.707106781	1.1391	0.4675
1.1	1.7321	0.707106781	0.2083	0.8276
1.2	2.0368	0.707106781	0.1092	0.9015
1.3	2.4281	0.707106781	0.0203	0.9801
1.41421356	3	0.707113562	0	1
1.6	4.2768	0.625	0.0516	0.9509
1.9	7.5161	0.526315789	0.6453	0.6078
2.2	12.7128	0.454545454	2.9849	0.2509
2.5	20.5313	0.4	9.7246	0.0932
2.8	31.7328	0.357142857	26.1217	0.0369
3.1	47.1761	0.322580645	61.7474	0.0159
3.4	67.8168	0.294117647	132.9291	7.4666×10^{-3}
3.7	94.7081	0.27027027	266.1093	3.7438×10^{-3}
4	129	0.25	502.3266	1.9868×10^{-3}
4.3	171.9401	0.232558139	903.0469	1.1061×10^{-3}
4.6	224.8728	0.217391304	1557.5901	6.4161×10^{-4}
4.9	289.2401	0.204081632	2592.4229	3.8559×10^{-4}

Continued

v=5, N=27, a*=2.0000				
a	c	g	Rv(D)	Pv(D)
1	1.125	0.447213595	762.0031	1.3106×10^{-3}
1.1	1.183	0.447213595	333.8232	2.9867×10^{-3}
1.2	1.2592	0.447213595	152.7573	6.5038×10^{-3}
1.3	1.357	0.447213595	71.7259	0.0138
1.6	1.8192	0.447213595	7.0363	0.1244
1.9	2.629	0.447213595	0.1756	0.8506
2	3	0.447213595	0	1
2.2	3.9282	0.447213595	0.3403	0.7461
2.5	5.8828	0.4	2.8821	0.2576
2.8	8.6832	0.357142857	11.2011	0.082
3.1	12.544	0.322580645	31.589	0.0307
3.4	17.7042	0.294117647	74.982	0.0132
3.7	24.427	0.27027027	159.2199	6.2414×10^{-3}
4	33	0.25	312.1165	3.1937×10^{-3}
4.3	43.735	0.232558139	575.453	1.7347×10^{-3}
4.6	56.9682	0.217391304	1010.0666	9.8905×10^{-4}
4.9	73.06	0.204081632	1702.2156	5.8712×10^{-4}
v=10, N=149, a*=3.36358566				
a	c	g	Rv(D)	Pv(D)
1	1.0156	0.316227766	203268.8282	4.9196×10^{-6}
1.1	1.0229	0.316227766	94134.6208	1.0623×10^{-5}
1.2	1.0324	0.316227766	46477.9484	2.1515×10^{-5}
1.3	1.0446	0.316227766	24195.572	4.1328×10^{-5}
1.6	1.1024	0.316227766	4327.8557	2.3101×10^{-4}
1.9	1.2036	0.316227766	980.8164	1.0185×10^{-3}
2.2	1.366	0.316227766	251.1497	3.9659×10^{-3}
2.5	1.6104	0.316227766	65.33	0.0151

Continued

2.8	1.9604	0.316227766	14.767	0.0634
3.1	2.443	0.316227766	1.8777	0.3475
3.36358566	3	0.297300511	0	1
3.4	3.088	0.294117647	0.034	0.9615
3.7	3.9284	0.27027027	4.4491	0.1835
4	5	0.25	20.6482	0.0462
4.3	6.3419	0.232558139	57.6506	0.0171
4.6	7.996	0.217391304	128.8464	7.7014×10^{-3}
4.9	10.0075	0.204081632	253.4835	3.9295×10^{-3}

a* indicates exact SORD using CCD.

Table 2: Measure of rotatability for second order response surface designs using balanced incomplete block designs (See, Victorbabu and Surekha, 2015)

(v=3,b=3,r=2,k=2, $\lambda = 1$), N=19, a* = 1.18920711				
a	c	g	Rv(D)	Pv(D)
1	2.5	1	$5.209235209 \times 10^{-3}$	0.994800
1.1	2.7321	0.909090909	$2.405163647 \times 10^{-3}$	0.997600
1.18920711	3	0.840901446	0	1.000000
1.2	3.0368	0.833333333	$6.580597223 \times 10^{-5}$	0.999900
1.3	3.42805	0.769230769	0.011886036	0.988300
1.6	5.2768	0.625	0.636669281	0.636700
1.9	8.51605	0.526315789	4.288667252	0.189100
2.2	13.7128	0.454545454	18.26928236	0.051900
2.5	21.53125	0.447213595	23.87124893	0.040200
2.8	32.7328	0.447213595	25.72475955	0.037400
3.1	48.17605	0.447213595	26.87013539	0.035900
3.4	68.8168	0.447213595	27.59913264	0.035000
3.7	95.70805	0.447213595	28.07744578	0.034400

Continued

4	130	0.447213595	28.40040383	0.034000
4.3	172.94005	0.447213595	28.62423443	0.033800
4.6	225.8728	0.447213595	28.78304771	0.033600
4.9	290.24005	0.447213595	28.89812235	0.033400
(v=15,b=15,r=7,k=7,$\lambda = 3$), N=991, a* = 2.82842713				
a	c	g	Rv(D)	Pv(D)
1	2.3438	1	$3.018689892 \times 10^{-3}$	0.997
1.1	2.3486	0.909090909	$6.330214105 \times 10^{-3}$	0.9937
1.2	2.3549	0.833333333	0.012335027	0.9878
1.3	2.3631	0.769230769	0.022541456	0.978
1.6	2.4016	0.625	0.099085858	0.9098
1.9	2.4691	0.526315789	0.280739268	0.7808
2.2	2.5774	0.454545454	0.498659245	0.6673
2.5	2.7402	0.4	0.430313296	0.6991
2.8	2.9736	0.357142857	$8.555997594 \times 10^{-3}$	0.9915
2.82842713	3	0.353556781	0	1
3.1	3.2953	0.322580645	1.787090102	0.3588
3.4	3.7254	0.294117647	16.0102761	0.0588
3.7	4.2856	0.27027027	68.06228677	0.0145
4	5	0.25	207.3656233	4.7993×10^{-3}
4.3	5.8946	0.232558139	517.3747802	1.9291×10^{-3}
4.6	6.9974	0.217391304	1127.210533	8.8636×10^{-4}
4.9	8.3383	0.204081632	2225.881372	4.4906×10^{-4}

a* indicates exact SORD using BIBD.

Table 3: Measure of rotatability for second order response surface designs using pairwise balanced designs (See, Victorbabu and Surekha, 2013)

(v=9,b=11,r=5,k1=5,k2=4,k3=3,$\lambda = 2$), N=195, a*=1.68179283				
a	c	g	Rv(D)	Pv(D)
1	2.5625	1	3.1354×10^{-3}	0.9969
1.1	2.5915	0.909090909	5.6478×10^{-3}	0.9944
1.2	2.6296	0.833333333	8.8841×10^{-3}	0.9912
1.3	2.6785	0.769230769	0.012	0.9882
1.6	2.9096	0.625	3.8494×10^{-3}	0.9962
1.68179283	3	0.594601022	0	1
1.9	3.3145	0.526315789	0.1254	0.8886
2.2	3.9641	0.454545454	2.3218	0.301
2.5	4.9414	0.4	14.8059	0.0633
2.8	6.3416	0.357142857	59.1309	0.0166
3.1	8.272	0.322580645	179.2755	5.5471×10^{-3}
3.4	10.8521	0.294117647	453.6589	2.1995×10^{-3}
3.7	14.2135	0.27027027	1011.6833	9.8748×10^{-4}
4	18.5	0.25	2056.1336	4.8611×10^{-4}
4.3	23.8675	0.232558139	3892.5594	2.5683×10^{-4}
4.6	30.4841	0.217391304	6966.8114	1.4352×10^{-4}
4.9	38.53	0.204081632	11912.0373	8.3942×10^{-5}
(v=14,b=15,r=7,k1=7,k2=6,$\lambda = 3$), N=989, a*=2.82842713				
a	c	g	Rv(D)	Pv(D)
1	2.3438	1	3.4084×10^{-3}	0.9966
1.1	2.3486	0.909090909	7.1474×10^{-3}	0.9929
1.2	2.3549	0.833333333	0.01393	0.9863
1.3	2.3631	0.769230769	0.0255	0.9752
1.6	2.4016	0.625	0.1119	0.8994

Continued

1.9	2.4691	0.526315789	0.317	0.7593
2.2	2.5774	0.454545454	0.563	0.6398
2.5	2.7402	0.4	0.4859	0.673
2.8	2.9736	0.357142857	9.6605×10^{-3}	0.9904
2.82842713	3	0.35556781	0	1
3.1	3.2953	0.322580645	2.0178	0.3314
3.4	3.7254	0.294117647	18.0771	0.0524
3.7	4.2856	0.27027027	76.8488	0.0128
4	5	0.25	234.1357	4.2529×10^{-3}
4.3	5.8946	0.232558139	584.1657	1.7089×10^{-3}
4.6	6.9974	0.217391304	1272.7287	7.8510×10^{-4}
4.9	8.3383	0.204081632	2513.2333	3.9774×10^{-4}

a* indicates exact SORD using PBD.

Table 4: Measure of rotatability for second order response surface designs using symmetrical unequal block arrangements with two unequal block sizes (See, Victorbabu and Surekha, 2013)

(v=6,b=11,r=7,k1=3,k2=4,b1=2,b2=9,$\lambda = 4$), N=189, a*=2.51486686				
a	c	g	Rv(D)	Pv(D)
1	1.7813	1	0.0355	0.9657
1.1	1.7958	0.909090909	0.0717	0.9331
1.2	1.8148	0.833333333	0.1328	0.8827
1.3	1.8393	0.769230769	0.2278	0.8145
1.6	1.9548	0.625	0.7514	0.571
1.9	2.1573	0.526315789	1.315	0.432
2.2	2.4821	0.454545454	0.9785	0.5054
2.5	2.9707	0.4	4.9234×10^{-3}	0.9951
2.51486686	3	0.397630124	0	1

Continued

2.8	3.6708	0.357142857	3.4795	0.2232
3.1	4.636	0.322580645	25.2093	0.0382
3.4	5.9261	0.294117647	91.9907	0.0108
3.7	7.6068	0.27027027	249.3299	3.9947×10^{-3}
4	9.75	0.25	569.4011	1.7532×10^{-3}
4.3	12.4338	0.232558139	1161.6785	8.6008×10^{-4}
4.6	15.7421	0.217391304	2186.6155	4.5712×10^{-4}
4.9	19.765	0.204081632	3872.8042	2.5814×10^{-4}
(v=12,b=15,r=7,k1=4,k2=6,b1=3,b2=12, $\lambda = 3$), N=505, a*=2.37841423				
a	c	g	Rv(D)	Pv(D)
1	2.3542	1	4.4155×10^{-3}	0.9956
1.1	2.3638	0.909090909	9.0541×10^{-3}	0.991
1.2	2.3765	0.833333333	0.0171	0.9832
1.3	2.3928	0.769230769	0.0301	0.9708
1.6	2.4699	0.625	0.1085	0.9022
1.9	2.6048	0.526315789	0.1999	0.8334
2.2	2.8214	0.454545454	0.1025	0.9071
2.37841423	3	0.420450723	0	1
2.5	3.1471	0.4	0.1391	0.8779
2.8	3.6139	0.357142857	4.0451	0.1982
3.1	4.2573	0.322580645	24.6694	0.039
3.4	5.1174	0.294117647	91.6793	0.0108
3.7	6.2378	0.27027027	260.5593	3.8232×10^{-3}
4	7.6667	0.25	623.3944	1.6016×10^{-3}
4.3	9.4558	0.232558139	1322.5936	7.5552×10^{-4}
4.6	11.6614	0.217391304	2568.6011	3.8917×10^{-4}
4.9	14.3433	0.204081632	4662.386786	2.1444×10^{-4}

a* indicates exact SORD using SUBA with two unequal block sizes.

Table 5: Values of measure of rotatability for second order response surface designs using pair of PBIBD (See, Victorbabu and Surekha, 2014)

D1=(v=6,b1=4,r1=2,k1=3,λ_{11}=1,λ_{12}=1), D2=(v=6,b2=3,r2=1,k2=2,21=0,22=1), N=44, a* = 1.1892				
a	c	g	Rv(D)	Pv(D)
1.1	2.73	0.9091	2.5980×10^{-3}	0.9974
1.1892	3	0.8409	0	1
1.2	3.04	0.8333	7.1082×10^{-5}	0.9999
1.3	3.43	0.7692	0.0128	0.9873
1.6	5.28	0.625	0.6164	0.6186
1.9	8.52	0.5263	4.6325	0.1775
2.2	13.71	0.4545	19.7339	0.0482
2.5	21.53	0.4	62.9517	0.0156
2.8	32.73	0.378	106.747	9.2810×10^{-3}
3.1	48.18	0.378	111.4998	8.8889×10^{-3}
3.4	68.82	0.378	114.5248	8.6561×10^{-3}
3.7	95.71	0.378	116.5096	8.5099×10^{-3}
4	130	0.378	117.8498	8.4140×10^{-3}
4.3	172.94	0.378	118.7786	8.3487×10^{-3}
4.6	225.87	0.378	119.4376	8.3031×10^{-3}
4.9	290.24	0.378	119.9151	8.2703×10^{-3}
D1=(v=12,b1=8,r1=4,k1=6,λ_{11}=2,λ_{12}=0), D2=(v=12,b2=6,r2=1,k2=2,λ_{21}=0, λ_{22}=1), N=280, a* = 2.0000				
a	c	g	Rv(D)	Pv(D)
1.3	2.18	0.7692	0.0532	0.9495
1.6	2.41	0.625	0.1012	0.9081
1.9	2.81	0.5263	0.0238	0.9767
2	3	0.5	0	1

Continued

2.2	3.46	0.4545	0.2614	0.7928
2.5	4.44	0.4	3.5944	0.2178
2.8	5.84	0.3571	17.4749	0.0541
3.1	7.77	0.3226	56.8674	0.0173
3.4	10.35	0.2941	148.1967	6.7026×10^{-3}
3.7	13.71	0.2703	334.9262	2.9768×10^{-3}
4	18	0.25	685.1211	1.4575×10^{-3}
4.3	23.37	0.2326	1301.3406	7.6785×10^{-4}
4.6	29.98	0.2174	2333.2519	4.2840×10^{-4}
4.9	38.03	0.2041	3993.4131	2.5035×10^{-4}

a* indicates exact SORD using SUBA with two unequal block sizes.

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Estimation of P(Y < X) based on Records for Kumaraswamy-Exponential Distribution

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ABSTRACT

In this paper, the problem of estimation of $R = P(Y < X)$ based on record values, when X and Y are two independent Kumaraswamy exponential (Kw-E) distributions is considered. The maximum likelihood (ML) estimator of R is obtained. Asymptotic distributions based on ML estimators are also obtained. Monte Carlo simulation is performed to study the behaviour of different estimators.

Key words and Phrases: Record values, Kumaraswamy exponential distribution, Maximum likelihood estimator, Exponential distribution.

1 Introduction

If X is the strength of a system which is subjected to a stress Y, then $R = P(Y < X)$ is a measure of system performance. The system fails if and only if at any time the applied stress is greater than its strength. Inference on R was carried out by several authors for the majority of common distribution families. Rezaei et al. (2010) and Wong (2012) made inferences about R for generalized pareto distributions. Baklizi (2008) estimated R when both stress and strength variables

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have two parameter exponential distribution. Estimation of R when Stress and strength are gamma distributed was considered by Krishnamoorthy et al. (2008) and Huang et al. (2012). Kundu and Gupta (2005) considered the problem of estimation of $P(Y < X)$, when X and Y are independent generalized exponential distribitions.

Let $\{X_i, i \geq 1\}$ be a sequence of independent and identically distributed (iid) random variables having an absolutely continuous cumulative distribution function (cdf) F(x) and probability density function (pdf) f(x). An observation X_j is called an upper record if $X_j > X_i$ for every $i < j$. An analogous definition deals with lower record values. In a number of situations only observations that exceed or only those that fall below the current extreme value are recorded. Examples include meterology, hydrology, athletic events and mining. Chandler (1952) introduced and studied some properties of record values. Useful surveys are given in Ahsanullah (2004) and Arnold et al. (1998).

In this paper, our interest is to estimate R, for Kumaraswamy-exponential distribution (Kw-E) based on upper record values. The cdf and pdf of Kw-E distribution are respectively given by

$$F(x) = 1 - (1 - (1 - \exp(-x))^{\alpha})^{\beta}, x > 0 \tag{1.1}$$

and

$$f(x) = \alpha\beta \exp(-x)(1 - \exp(-x))^{\alpha-1}(1 - (1 - \exp(-x))^{\alpha})^{\beta-1}, x > 0. \tag{1.2}$$

where $\alpha > 0$, $\beta > 0$ are two shape parameters.
Through out the paper we use the notation Kw-E (α, β) to denote Kumaraswamy-Exponential distribution.

In this paper we consider the problem of estimation of $R = P(Y < X)$ when X and Y follow Kw-E distributions. In section 2 and 3, the estimation of R are discussed when common shape parameter α is unknown and known respectively. In section 4, the estimation of R is studied in the general case when all the parameters are assumed unknown and different. Section 5 is devoted to a simulation study.

2 Estimation of R with common unknown shape parameter α

Let X and Y be two independent random variables follow Kw-E (α, β_1) and Kw-E (α, β_2) distributions respectively. Then $R = P(Y < X)$ is given by

$$
\begin{aligned}
R &= P(Y < X) \\
&= \int_0^{\infty}\int_0^{x} \alpha\beta_1 \exp(-x)(1-\exp(-x))^{\alpha-1}(1-(1-\exp(-x))^{\alpha})^{\beta_1-1} \\
&\quad \times\alpha\beta_2 \exp(-y)(1-\exp(-y))^{\alpha-1}(1-(1-\exp(-y))^{\alpha})^{\beta_2-1} \\
&= \frac{\beta_2}{\beta_1+\beta_2}.
\end{aligned}
$$

2.1 The maximum likelihood estimator of R

Let $R_1, R_2, ..., R_n$ be the first n upper record values arising from Kw-E (α, β_1) distribution and $S_1, S_2, ..., S_m$ be the first m upper record values arising from Kw-E (α, β_2) distribution. Then the likelihood functions are given by

$$
\begin{aligned}
L_1 &= L(\alpha, \beta_1 | r_0, r_1, ..., r_n) = f(r_n)\prod_{i=1}^{n-1}\frac{f(r_i)}{1-F(r_i)} \\
&= \alpha^n \beta_1^n (1-(1-\exp(-r_n))^{\alpha})^{\beta_1} \\
&\quad \times\prod_{i=1}^{n} \exp(-r_i)(1-\exp(-r_i))^{\alpha-1}(1-(1-\exp(-r_i))^{\alpha})^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
L_2 &= L(\alpha, \beta_2 | s_0, s_1, ..., s_m) = f(s_m)\prod_{j=1}^{m-1}\frac{f(s_j)}{1-F(s_j)} \\
&= \alpha^m \beta_2^m (1-(1-\exp(-s_m))^{\alpha})^{\beta_2} \\
&\quad \times\prod_{j=1}^{m} \exp(-s_j)(1-\exp(-s_j))^{\alpha-1}(1-(1-\exp(-s_j))^{\alpha})^{-1}.
\end{aligned}
$$

Then the joint loglikelihood function of α, β_1 and β_2 is given by

$$\begin{aligned} L &= (m+n)\log\alpha + n\log\beta_1 + m\log\beta_2 - \sum_{i=1}^{n} r_i - \sum_{j=1}^{m} s_j \\ &\quad +(\alpha-1)\left(\sum_{i=1}^{n}\log(1-\exp(-r_i)) + \sum_{j=1}^{m}\log(1-\exp(-s_j))\right) \\ &\quad -\sum_{i=1}^{n}\log(1-(1-\exp(-r_i))^{\alpha} - \sum_{j=1}^{m}\log(1-\exp(-s_j))^{\alpha}) \\ &\quad +\beta_1\log(1-(1-\exp(-r_n))^{\alpha}) + \beta_2\log(1-(1-\exp(-s_m))^{\alpha}). \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial L}{\partial \alpha} &= \frac{n+m}{\alpha} + \sum_{i=1}^{n}\log(1-\exp(-r_i)) + \sum_{i=1}^{n}\frac{(1-\exp(-r_i))^{\alpha}\log(1-\exp(-r_i))}{1-(1-\exp(-r_i))^{\alpha}} \\ &\quad +\sum_{j=1}^{m}\log(1-\exp(-s_j)) + \sum_{j=1}^{m}\frac{(1-\exp(-s_j))^{\alpha}\log(1-\exp(-s_j))}{1-(1-\exp(-s_j))^{\alpha}} \\ &\quad -\beta_1\left(\frac{(1-\exp(-r_n))^{\alpha}\log(1-\exp(-r_n))}{1-(1-\exp(-r_n))^{\alpha}}\right) \\ &\quad -\beta_2\left(\frac{(1-\exp(-s_m))^{\alpha}\log(1-\exp(-s_m))}{1-(1-\exp(-s_m))^{\alpha}}\right), \end{aligned}$$

$$\frac{\partial L}{\partial \beta_1} = \frac{n}{\beta_1} + \log(1-(1-\exp(-r_n))^{\alpha})$$

and

$$\frac{\partial L}{\partial \beta_2} = \frac{m}{\beta_2} + \log(1-(1-\exp(-s_m))^{\alpha}).$$

The maximum likelihood (ML) estimators $\hat{\alpha}, \hat{\beta_1}$ and $\hat{\beta_2}$ of the parameters of α, β_1 and β_2 respectively can then be obtained as the solution of the following equations.

$$\frac{\partial L}{\partial \alpha} = 0 \qquad (2.1)$$

$$\frac{\partial L}{\partial \beta_1} = 0 \qquad (2.2)$$

and

$$\frac{\partial L}{\partial \beta_2} = 0 \qquad (2.3)$$

Thus from (2.2) and (2.3) we have

$$\hat{\beta}_1 = \frac{-n}{\log(1-(1-\exp(-r_n))^{\alpha})} \tag{2.4}$$

and

$$\hat{\beta}_2 = \frac{-m}{\log(1-(1-\exp(-s_m))^{\alpha})} \tag{2.5}$$

Substituting (2.4) and (2.5) into (2.1), gives

$$\begin{aligned}
&\frac{n+m}{\alpha} + \sum_{i=1}^{n} \log(1-\exp(-r_i)) + \sum_{i=1}^{n} \frac{(1-\exp(-r_i))^{\alpha} \log(1-\exp(-r_i))}{1-(1-\exp(-r_i))^{\alpha}} \\
&+ \sum_{j=1}^{m} \log(1-\exp(-s_j)) + \sum_{j=1}^{m} \frac{(1-\exp(-s_j))^{\alpha} \log(1-\exp(-s_j))}{1-(1-\exp(-s_j))^{\alpha}} \\
&+ \frac{n}{\log(1-(1-\exp(-r_n))^{\alpha})} \left(\frac{(1-\exp(-r_n))^{\alpha} \log(1-\exp(-r_n))}{1-(1-\exp(-r_n))^{\alpha}} \right) \\
&+ \frac{m}{\log(1-(1-\exp(-s_m))^{\alpha})} \left(\frac{(1-\exp(-s_m))^{\alpha} \log(1-\exp(-s_m))}{1-(1-\exp(-s_m))^{\alpha}} \right) = 0.
\end{aligned} \tag{2.6}$$

Let $\hat{\alpha}$ be the estimator of α obtained by solving the nonlinear equation (2.6) with respect to α and then by using equations (2.4) and (2.5), the ML estimators of β_1 and β_2 will be given by

$$\hat{\beta}_1 = \frac{-n}{\log(1-(1-\exp(-r_n))^{\hat{\alpha}})}$$

and

$$\hat{\beta}_2 = \frac{-m}{\log(1-(1-\exp(-s_m))^{\hat{\alpha}})}.$$

Once the estimators $\hat{\beta}_1$ and $\hat{\beta}_2$ of β_1 and β_2 are obtained and by the invariance property of the ML estimators, the ML estimator of R becomes

$$\begin{aligned}
\hat{R} &= \frac{\hat{\beta}_2}{\hat{\beta}_1 + \hat{\beta}_2} \\
&= \frac{m \log(1-(1-\exp(-r_n))^{\hat{\alpha}})}{n \log(1-(1-\exp(-s_m))^{\hat{\alpha}}) + m \log(1-(1-\exp(-r_n))^{\hat{\alpha}})}.
\end{aligned} \tag{2.7}$$

3 Estimation of R with known shape parameter α

In this section, the problem of estimation of R when α is known is considered. That is, it is assumed that $R_1, R_2, ..., R_n$ be the first n upper record values arising from Kw-E (α, β_1) distribution and $S_1, S_2, ..., S_m$ be the first m upper record values arising from Kw-E (α, β_2) distribution with α known.

3.1 The Maximum Likelihood Estimator of R

Based on the equations (2.2) and (2.3), the ML estimators of β_1 and β_2 when α is known are given by,

$$\hat{\beta}_1 = \frac{-n}{\log(1-(1-\exp(-r_n))^{\alpha})} \tag{3.1}$$

and

$$\hat{\beta}_2 = \frac{-m}{\log(1-(1-\exp(-s_m))^{\alpha})}. \tag{3.2}$$

Thus the ML estimator of R becomes

$$\hat{R} = \frac{m\log(1-(1-\exp(-r_n))^{\alpha})}{n\log(1-(1-\exp(-s_m))^{\alpha}) + m\log(1-(1-\exp(-r_n))^{\alpha})}.$$

3.2 Asymptotic Distribution

Based on the asymptotic properties under general conditions of the ML estimators $\hat{\alpha}$, $\hat{\beta}_1$ and $\hat{\beta}_2$ the asymptotic distribution of the ML estimators immediately follows from the Fisher information matrix of α, β_1 and β_2 (Lehmann, 1999) that is, when $n \to \infty$,$m \to \infty$ and $\frac{n}{m} \to p$, $0 < p < 1$

$$\left(\sqrt{n}(\hat{\beta}_1 - \beta_1), \sqrt{m}(\hat{\beta}_2 - \beta_2), \sqrt{n}(\hat{\alpha} - \alpha)\right) \to N_3(0, \Sigma_3),$$

where

$$\Sigma_3 = I^{-1}(\Omega) = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}^{-1}.$$

The matrix $I(\Omega)$ is the Fisher information matrix of the parameter vector $\Omega = \{\beta_1, \beta_2, \alpha\}$ and ij^{th} element is given by

$$I_{ij} = \frac{\partial^2 L}{\partial \omega_i \partial \omega_j}, i, j = 1, 2,$$

with

$$\begin{aligned}
I_{11} &= \frac{-n}{\beta_1^2}, \\
I_{12} &= I_{21} = 0, \\
I_{13} &= I_{31} = \frac{(1-\exp(-r_n))^\alpha \log(1-\exp(-r_n))}{1-(1-\exp(-r_n))^\alpha}, \\
I_{22} &= \frac{-m}{\beta_2^2}, \\
I_{23} &= I_{32} = \frac{(1-\exp(-s_m))^\alpha \log(1-\exp(-s_m))}{1-(1-\exp(-s_m))^\alpha}
\end{aligned}$$

and

$$\begin{aligned}
I_{33} = {} & \frac{-(m+n)}{\alpha^2} - \sum_{i=1}^{n} \frac{[\log(1-\exp(-r_i))]^2 (1-\exp(-r_i))^\alpha}{[1-(1-\exp(-r_i))^\alpha]^2} \\
& - \sum_{j=1}^{m} \frac{[\log(1-\exp(-s_j))]^2 (1-\exp(-s_j))^\alpha}{[1-(1-\exp(-s_j))^\alpha]^2} \\
& + \beta_1 \frac{[\log(1-\exp(-r_n))]^2 (1-\exp(-r_n))^\alpha}{[1-(1-\exp(-r_n))^\alpha]^2} \\
& + \beta_2 \frac{[\log(1-\exp(-s_m))]^2 (1-\exp(-s_m))^\alpha}{[1-(1-\exp(-s_m))^\alpha]^2}.
\end{aligned}$$

4 Estimation of R in the general case

Let X and Y be two independent Kw-E (α_1, β_1) and Kw-E (α_2, β_2) random variables respectively. Then the reliability $R = P(Y < X)$ is given by

$$\begin{aligned}
R &= P(Y < X) \\
&= \int_0^\infty \int_0^x \alpha_1 \beta_1 \exp(-x)(1-\exp(-x))^{\alpha_1 - 1}(1-(1-\exp(-x))^{\alpha_1})^{\beta_1 - 1} \\
&\quad \times \alpha_2 \beta_2 \exp(-y)(1-\exp(-y))^{\alpha_2 - 1}(1-(1-\exp(-y))^{\alpha_2})^{\beta_2 - 1} \\
&= 1 - \alpha_1 \beta_1 \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{k+l} \Gamma(\beta_1) \Gamma(\beta_2 + 1)}{\Gamma(\beta_1 - k) \Gamma(\beta_2 - l + 1) k! l! (\alpha_1(k+1) + \alpha_2 l)}.
\end{aligned}$$

4.1 The maximum likelihood estimator of R

Let $R_1, R_2, ..., R_n$ be the first n upper record values arising from Kw-E (α_1, β_1) distribution and $S_1, S_2, ..., S_m$ be the first m upper record values arising from Kw-E (α_2, β_2) distribution. The log-likelihood function of $\alpha_1, \alpha_2, \beta_1$ and β_2 is given by

$$\begin{aligned} L &= n\log\alpha_1 + m\log\alpha_2 + n\log\beta_1 + m\log\beta_2 + (\alpha_1-1)\sum_{i=1}^{n}\log(1-\exp(-r_i)) \\ &\quad +(\alpha_2-1)\sum_{j=1}^{m}\log(1-\exp(-s_j)) + \beta_1\log(1-(1-\exp(-r_n))^{\alpha_1}) \\ &\quad -\sum_{i=1}^{n} r_i - \sum_{j=1}^{m} s_j + \beta_2\log(1-(1-\exp(-s_m))^{\alpha_2}) \\ &\quad -\sum_{i=1}^{n}\log(1-(1-\exp(-r_i))^{\alpha_1}) - \sum_{j=1}^{m}\log(1-(1-\exp(-s_j))^{\alpha_2}) \end{aligned}$$

$$\begin{aligned} \frac{\partial L}{\partial \alpha_1} &= \frac{n}{\alpha_1} + \sum_{i=1}^{n}\log(1-\exp(-r_i)) + \sum_{i=1}^{n}\frac{(1-\exp(-r_i))^{\alpha_1}\log(1-\exp(-r_i))}{1-(1-\exp(-r_i))^{\alpha_1}} \\ &\quad -\beta_1\left(\frac{(1-\exp(-r_n))^{\alpha_1}\log(1-\exp(-r_n))}{1-(1-\exp(-r_n))^{\alpha_1}}\right), \\ \frac{\partial L}{\partial \alpha_2} &= \frac{m}{\alpha_2} + \sum_{j=1}^{m}\log(1-\exp(-s_j)) + \sum_{j=1}^{m}\frac{(1-\exp(-s_j))^{\alpha_2}\log(1-\exp(-s_j))}{1-(1-\exp(-s_j))^{\alpha_2}} \\ &\quad -\beta_2\left(\frac{(1-\exp(-s_m))^{\alpha_2}\log(1-\exp(-s_m))}{1-(1-\exp(-s_m))^{\alpha_2}}\right), \\ \frac{\partial L}{\partial \beta_1} &= \frac{n}{\beta_1} + \log(1-(1-\exp(-r_n))^{\alpha_1}) \\ \text{and} & \\ \frac{\partial L}{\partial \beta_2} &= \frac{m}{\beta_2} + \log(1-(1-\exp(-s_m))^{\alpha_2}). \end{aligned}$$

The estimators $\hat{\alpha_1}$, $\hat{\alpha_2}$, $\hat{\beta_1}$ and $\hat{\beta_2}$ of the parameters α_1, α_2, β_1 and β_2 respectively are obtained by solving the following equations,

$$\frac{\partial L}{\partial \alpha_1} = 0$$

$$\frac{\partial L}{\partial \alpha_2} = 0$$

$$\frac{\partial L}{\partial \beta_1} = 0 \tag{4.1}$$

and

$$\frac{\partial L}{\partial \beta_2} = 0 \tag{4.2}$$

From 4.1 and 4.2 we have

$$\hat{\beta_1} = \frac{-n}{\log[1 - (1 - \exp(-r_n))^{\hat{\alpha_1}}]} \tag{4.3}$$

and

$$\hat{\beta_2} = \frac{-m}{\log[1 - (1 - \exp(-s_m))^{\hat{\alpha_2}}]}, \tag{4.4}$$

where $\hat{\alpha_1}$ and $\hat{\alpha_2}$ are the solutions of the following two non linear equations

$$\frac{n}{\hat{\alpha_1}} + \sum_{i=1}^{n} \log(1 - \exp(-r_i)) - \beta_1 \left(\frac{(1 - \exp(-r_n))^{\alpha_1} \log(1 - \exp(-r_n))}{1 - (1 - \exp(-r_n))^{\alpha_1}} \right) + \sum_{i=1}^{n} \frac{(1 - \exp(-r_i))^{\alpha_1} \log(1 - \exp(-r_i))}{1 - (1 - \exp(-r_i))^{\alpha_1}} = 0$$

and

$$\frac{m}{\hat{\alpha_2}} + \sum_{j=1}^{m} \log(1 - \exp(-s_j)) - \beta_2 \left(\frac{(1 - \exp(-s_m))^{\alpha_2} \log(1 - \exp(-s_m))}{1 - (1 - \exp(-s_m))^{\alpha_2}} \right) + \sum_{j=1}^{m} \frac{(1 - \exp(-s_j))^{\alpha_2} \log(1 - \exp(-s_j))}{1 - (1 - \exp(-s_j))^{\alpha_2}} = 0$$

Thus the ML estimator of R is obtained as

$$\hat{R} = 1 - \hat{\alpha_1}\hat{\beta_1} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{k+l}\Gamma(\hat{\beta_1} - 1)\Gamma(\hat{\beta_2})}{\Gamma(\hat{\beta_1} - k - 1)\Gamma(\hat{\beta_2} - l)k!l!(\hat{\alpha_1}(k+1) + \hat{\alpha_2} l)}.$$

4.2 Asymptotic Distribution

Based on the asymptotic properties under general conditions of the ML estimators $\hat{\alpha_1}$, $\hat{\alpha_2}$, $\hat{\beta_1}$ and $\hat{\beta_2}$, the asymptotic distribution of the ML estimators immediately follows from the Fisher information matrix of α_1, α_2, β_1 and β_2 (Lehmann, 1999), that is when $n \to \infty$,$m \to \infty$ and $\frac{n}{m} \to p$, $0 < p < 1$

$$\left(\sqrt{n}(\hat{\beta_1} - \beta_1), \sqrt{n}(\hat{\alpha_1} - \alpha_1), \sqrt{m}(\hat{\beta_2} - \beta_2), \sqrt{m}(\hat{\alpha_2} - \alpha_2)\right) \to N_4(0, \Sigma_4),$$

where

$$\Sigma_4 = I^{-1}(\Omega) = \begin{bmatrix} I_{11}\ I_{12}\ I_{13}\ I_{14} \\ I_{21}\ I_{22}\ I_{23}\ I_{24} \\ I_{31}\ I_{32}\ I_{33}\ I_{34} \\ I_{41}\ I_{42}\ I_{43}\ I_{44} \end{bmatrix}^{-1},$$

where

$$I_{ij} = \frac{\partial^2 L}{\partial \omega_i \partial \omega_j}, i, j = 1, 2, 3, 4$$

and $\Omega = \{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ with

$$\begin{aligned}
I_{11} &= \frac{-n}{\alpha_1^2} + \beta_1 \frac{[\log(1-\exp(-r_n))]^2 (1-\exp(-r_n))^{\alpha_1}}{[1-(1-\exp(-r_n))^{\alpha_1}]^2} \\
&\quad - \sum_{i=1}^{n} \frac{[\log(1-\exp(-r_i))]^2 (1-\exp(-r_i))^{\alpha_1}}{[1-(1-\exp(-r_i))^{\alpha_1}]^2}, \\
I_{12} &= 0, \\
I_{13} &= \frac{(1-\exp(-r_n))^{\alpha_1} \log(1-\exp(-r_n))}{1-(1-\exp(-r_n))^{\alpha_1}}, \\
I_{14} &= 0, \\
I_{22} &= \frac{-m}{\alpha_2^2} + \beta_2 \frac{[\log(1-\exp(-s_m))]^2 (1-\exp(-s_m))^{\alpha_2}}{[1-(1-\exp(-s_m))^{\alpha_2}]^2} \\
&\quad - \sum_{j=1}^{m} \frac{[\log(1-\exp(-s_j))]^2 (1-\exp(-s_j))^{\alpha_2}}{[1-(1-\exp(-s_j))^{\alpha_2}]^2}, \\
I_{23} &= 0, \\
I_{24} &= \frac{(1-\exp(-s_m))^{\alpha_2} \log(1-\exp(-s_m))}{1-(1-\exp(-s_m))^{\alpha_2}}, \\
I_{33} &= \frac{-n}{\beta_1^2}, \\
I_{34} &= 0,
\end{aligned}$$

and

$$I_{44} = \frac{-m}{\beta_2^2}.$$

5 Simulation Study

In this section a Monte Carlo simulation is performed to study the behaviour of different estimators developed in this paper. For the simulation study we generate n record values from Kw-E (α, β_1) and m record values from Kw-E (α, β_2) for different combinations of n and m and for different values of α, β_1 and β_2 . We have obtained the estimator $\widehat{R}$, the average bias and mean square error of $\widehat{R}$ when the shape parameter α is known and unknown over 1000 replications, and are given in table1 and table2 respectively.

Table 1: The MLE $\widehat{R}$ of R, its average bias and mean square error $MSE(\widehat{R})$, when the shape parameter α is known $(\alpha = 1)$

(n,m)	$(\alpha, \beta_1, \beta_2)$	R	$\widehat{R}$	Bias	$MSE(\widehat{R})$
(5,5)	(1,0.5,0.5)	0.5000	0.5007	0.0007	0.0226
	(1,0.5,1)	0.6667	0.6551	-0.0115	0.0201
	(1,0.5,1.5)	0.7500	0.7342	-0.0157	0.0146
	(1,0.5,2)	0.800	0.7796	-0.0203	0.0118
	(1,0.5,3)	0.8571	0.8401	-0.0171	0.0084
(6,6)	(1,0.5,0.5)	0.5000	0.4988	-0.0011	0.0186
	(1,0.5,1)	0.6667	0.6535	-0.0131	0.0159
	(1,0.5,1.5)	0.7500	0.7332	-0.0167	0.0129
	(1,0.5,2)	0.8000	0.7843	-0.0156	0.0097
	(1,0.5,3)	0.8571	0.8436	-0.0134	0.006
(8,8)	(1,0.5,0.5)	0.5000	0.4955	-0.0045	0.0147
	(1,0.5,1)	0.6667	0.6568	-0.0098	0.0125
	(1,0.5,1.5)	0.7500	0.7353	-0.0146	0.0095
	(1,0.5,2)	0.8000	0.7858	-0.0142	0.0072
	(1,0.5,3)	0.8571	0.8452	-0.0119	0.005
(10,10)	(1,0.5,0.5)	0.5000	0.4949	-0.0051	0.0146
	(1,0.5,1)	0.6667	0.6589	-0.0077	0.0108
	(1,0.5,1.5)	0.7500	0.7465	-0.0034	0.0078
	(1,0.5,2)	0.8000	0.7958	-0.0042	0.006
	(1,0.5,3)	0.8571	0.8487	-0.0084	0.0039

Table 2: The MLE $\widehat{R}$ of R, its average bias and mean square error $MSE(\widehat{R})$, when the shape parameter α is unknown

(n,m)	$(\alpha, \beta_1, \beta_2)$	R	$\widehat{R}$	Bias	$MSE(\widehat{R})$
(5,5)	(1,0.5,0.5)	0.5000	0.4854	-0.0145	0.1626
	(1.5,1.5,1)	0.4000	0.3318	-0.0681	0.1044
	(1,1,1.5)	0.6000	0.6468	0.0468	0.0911
	(1.5,1.5,2)	0.5714	0.5757	0.0043	0.0647
	(1.5,2.5,3)	0.5454	0.5303	-0.0151	0.0184
(6,6)	(1,0.5,0.5)	0.5000	0.4973	-0.0026	0.147
	(1.5,1.5,1)	0.4000	0.3349	-0.0651	0.1034
	(1,1,1.5)	0.6000	0.6674	0.0674	0.1036
	(1.5,1.5,2)	0.5714	0.5649	-0.0065	0.0689
	(1.5,2.5,3)	0.5454	0.5229	-0.0225	0.0198
(8,8)	(1,0.5,0.5)	0.5000	0.4972	-0.0028	0.1039
	(1.5,1.5,1)	0.4000	0.2954	-0.1045	0.1089
	(1,1,1.5)	0.6000	0.7016	0.1016	0.1089
	(1.5,1.5,2)	0.5714	0.5913	0.0199	0.0808
	(1.5,2.5,3)	0.5454	0.5018	-0.0436	0.0302
(10,10)	(1,0.5,0.5)	0.5000	0.4785	-0.0214	0.0707
	(1.5,1.5,1)	0.4000	0.2601	-0.1398	0.1079
	(1,1,1.5)	0.6000	0.7562	0.1562	0.1146
	(1.5,1.5,2)	0.5714	0.6394	0.0679	0.0859
	(1.5,2.5,3)	0.5454	0.5075	-0.0379	0.0466

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