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CONTENTS

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A Discrete Generalization of Marshall-Olkin Scheme and its Application to Geometric Distribution

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ABSTRACT

In this paper, we introduce a new family of discrete distributions and study its properties. The new family is a generalization of discrete Marshall-Olkin family of distributions. In particular, we study discrete generalized Marshall-Olkin exponential (DGMOE) distribution in detail. Generalized geometric distribution and geometric distribution are sub-models of DGMOE distribution. We derive some basic distributional properties such as probability generating function, moments, hazard rate and quantiles of the DGMOE distribution. Estimation of the parameters are done using maximum likelihood method. A real data set is analyzed to illustrate the suitability of the proposed model.

Key words and Phrases: *Geometric distribution, Generalized geometric distribution, Generalized Marshall-Olkin family of distributions, Infinite divisibility, Maximum likelihood.*

1 Introduction

Modeling of count data is becoming popular due to its potential applications in areas like Insurance, Ecology, Public health etc. For this purpose, traditional models

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like Poisson, negative binomial and geometric were used. However, often it has been found that count data exhibits over-dispersion and corresponding distribution function shows long tail behavior. Hence there is further scope to modify/generalize these traditional models. This results us to devise ways of generalizing standard models, preserving the fundamental properties like unimodality, infinite divisibility and over dispersion/under dispersion of standard distribution.

Discretization of a continuous lifetime model is an interesting and intuitively appealing approach to derive a lifetime model corresponding to the continuous one. Chakraborty (2015) surveyed different methods for generating discrete analogues of continuous probability distributions. Out of these, one method is: if the underlying continuous life time X has the survival function $\bar{F}(x) = P(X > x)$, the probability mass function (pmf) of the discrete random variable associated with that continuous distribution can be written as

$$P(X = x) = p_x = \bar{F}(x) - \bar{F}(x+1); \quad x = 0, 1, 2, \ldots. \tag{1.1}$$

This method, applied to generate new distribution, has received much attention in the past. Nakagawa and Osaki (1975) derived discrete Weibull distribution, Roy (2004) analyzed the discrete Rayleigh distribution, Kemp (2008) examined the discrete halfnormal distribution and Krishna and Singh (2009) obtained the discrete Burr distribution. Gómez-Déniz (2010) developed a new generalization of the geometric distribution using Marshall-Olkin scheme and Gómez-Déniz and Calderin (2011) studied the discrete Lindely distribution. Nekoukhou et al. (2013) derived discrete generalized exponential distribution of a second type. Chakraborty and Chakravarty (2012, 2014, 2015) analyzed discrete versions of Gamma, Gumbel and power distributions.

Marshall and Olkin (1997) developed a simple method to add a parameter to a family of distributions. Let X_1, X_2,... be a sequence of independent and identically distributed random variables with survival function $\bar{F}(x)$. Suppose that N is independent of X_i's with geometric(α) distribution, that is, $P(N = n) = \alpha(1-\alpha)^{n-1}$, for n=1, 2, ... and $0 < \alpha < 1$. Then,

$$U_N = \min(X_1, X_2, ..., X_N) \tag{1.2}$$

has the survival function given by

$$S(x;\alpha) = \frac{\alpha \bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}, \quad 0<\alpha<1,\ \bar{\alpha}=1-\alpha,\ -\infty<x<\infty.$$

If $\alpha > 1$ and N is a geometric random variable with pmf $P(N=n) = \frac{1}{\alpha}(1-\frac{1}{\alpha})^{n-1}$, n =1, 2, ... then V_N=max$(X_1, X_2, ..., X_N)$ also has the same survival function,

$$S(x;\alpha) = \frac{\alpha \bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}, \quad \alpha>1,\ -\infty<x<\infty.$$

Hence, by combining, we obtain

$$S(x;\alpha) = \frac{\alpha \bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}, \quad \alpha>0,\ \bar{\alpha}=1-\alpha,\ -\infty<x<\infty. \tag{1.3}$$

Using $P(X=x) = S(x;\alpha) - S(x+1;\alpha)$, Gómez-Déniz (2010) developed a generalization of geometric distribution by putting $\bar{F}(x) = e^{-\lambda x}, \lambda > 0$ in (1.3), which is the survival function of exponential distribution. That is,

$$P(X=x) = \frac{\alpha\theta^x(1-\theta)}{(1-\bar{\alpha}\theta^{x+1})(1-\bar{\alpha}\theta^x)}, \quad x=0,1,2,... \tag{1.4}$$

where $\alpha > 0$, $\bar{\alpha} = 1-\alpha$, $\theta = e^{-\lambda}$, $0 < \theta < 1$. We call the distribution having pmf (1.4) as Generalized Geometric distribution and dented it as GG(α, θ). The corresponding distribution function of GG(α, θ) is

$$G(x;\alpha,\theta) = \frac{1-\theta^{x+1}}{1-\bar{\alpha}\theta^{x+1}}, \quad x=0,1,2,\dots. \tag{1.5}$$

Let the random variable X has geometric distribution with pmf

$$P(X=x) = pq^x, \quad x=0,\ 1,\ 2...; \quad 0<p<1,\ q=1-p.$$

Then the cumulative distribution function (cdf) of X is

$$F_X(x) = (1-q^{x+1}), \quad 0<q<1.$$

Chakraborty and Gupta (2015) studied the distribution of the random variable Y having cdf $[F_X(x)]^\alpha$, $\alpha > 0$. The pmf of Y is given by

$$P(Y=y) = \bar{F}_Y(y) - \bar{F}_Y(y+1) = (1-q^{y+1})^\alpha - (1-q^y)^\alpha, \quad y=0,1,2,... \tag{1.6}$$

Note that when $\alpha = 1$, (1.6) reduces to geometric distribution.

Also Bidram et al. (2016) introduced and studied a three parameter exponentiated generalized geometric distribution by exponentiating the GG(α, θ) distribution in (1.5). That is,

$$P(X = x) = \left[\frac{1-\theta^{x+1}}{1-\bar{\alpha}\theta^{x+1}}\right]^{\gamma} - \left[\frac{1-\theta^{x}}{1-\bar{\alpha}\theta^{x}}\right]^{\gamma}, \quad x = 0, 1, 2, \ldots,$$

where $\alpha > 0$, $\gamma > 0$ and $0 < \theta < 1$.

In this paper, we introduce discrete generalized Marshall-Olkin distribution. One member of this family, namely, discrete generalized Marshall-Olkin exponential (DGMOE) distribution is studied. It is an extension of generalized geometric distribution studied in Gómez-Déniz (2010). This distribution can also be considered as a discrete version of generalized Marshall-Olkin exponential distribution.

In Section 2, we introduce discrete generalized Marshall-Olkin distribution and study its properties. In particular, we consider DGMOE distribution in Section 3. It is shown that GG and geometric distributions are special case of DGMOE. The shape properties of pmf and hazard rate of DGMOE are discussed. The expression for moments and order statistics are obtained. In Section 4, we discuss the estimation of the parameters of DGMOE using the method maximum likelihood. Application of DGMOE distribution in modeling a real data set is presented in Section 5. It is shown that for the data DGMOE is better fit as compared to exponentiated generalized geometric distribution (Bidram et al. (2016)), exponentiated geometric distribution (Chakraborty and Gupta (2015)), Kumaraswamy-geometric distribution (Akinsete et al. (2014)), generalized geometric distribution (Gómez-Déniz (2010)) and geometric distribution. Concluding remarks are given in Section 6.

2 Discrete generalized Marshall-Olkin distribution

Jayakumar and Mathew (2008) proposed a generalization of Marshall-Olkin family of distributions by adding another parameter $\gamma > 0$, which is the resilience parameter, in the Marshall-Olkin scheme. Starting with a survival function $\bar{F}$, they proposed the two-parameter family of survival functions as follows:

$$\bar{G}_{\alpha,\gamma} = \left[\frac{\alpha\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}\right]^{\gamma}; \quad -\infty < x < \infty,\, 0 < \alpha < \infty,\, 0 < \gamma < \infty. \tag{2.1}$$

When $\alpha = 1$, we get $\bar{G}_{1,\gamma}(x) = \left[\bar{F}(x)\right]^{\gamma}$ and in particular, when $\alpha = \gamma = 1$, we get $\bar{G}_{1,1}(x) = \bar{F}(x)$.

The cumulative distribution function (cdf) of X is,

$$G_{\alpha,\gamma} = 1 - \left[\frac{\alpha\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}\right]^{\gamma}; \quad -\infty < x < \infty,\, 0 < \alpha < \infty,\, 0 < \gamma < \infty. \tag{2.2}$$

For $i = 1, 2, ..., \gamma$, where $\gamma > 1$ is an integer, if $T_{i1}, T_{i2}, ..., T_{iN}$ be a sequence of i.i.d random variables with survival function $\bar{F}(x)$, and

1. If N has a geometric distribution with parameters $\alpha (0 < \alpha \leq 1)$ independent of T_{ij}'s, then $\min_{1\leq i\leq\gamma}\{\min(T_{i1}, T_{i2}, ..., T_{iN})\}$ is distributed as generalized Marshall-Olkin distribution.

2. If N has a geometric distribution with parameters $\frac{1}{\alpha}$, $(\alpha > 1)$ independent of T_{ij}'s, then $\min_{1\leq i\leq\gamma}\{\max(T_{i1}, T_{i2}, ..., T_{iN})\}$ is distributed as generalized Marshall-Olkin distribution.

Now, if the random variable X with survival function $\bar{F}(x)$ is continuous, by applying (1.1) to the family of cdf (2.2), we obtain new discrete distribution with the pmf $g(x)$ given by

$$g(x;\alpha,\gamma) = P(X = x) = \left[\frac{\alpha\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}\right]^{\gamma} - \left[\frac{\alpha\bar{F}(x+1)}{1-\bar{\alpha}\bar{F}(x+1)}\right]^{\gamma}; \quad 0 < \alpha < \infty,\, 0 < \gamma < \infty. \tag{2.3}$$

Note that, when $\gamma = 1$, the distribution with pmf (2.3) reduces to discrete Marshall-Olkin distribution discussed in Supanekar and Shirke (2015).

2.1 Distribution function and Hazard rate

The survival function of the discrete random variable having the pmf (2.3) is given by

$$\bar{G}(x;\alpha,\gamma) = \left[\frac{\alpha\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}\right]^{\gamma}; \quad 0 < \alpha < \infty,\, 0 < \gamma < \infty,\, x = 0, 1, 2, ...$$

Therefore,

$$\begin{aligned} G(x;\alpha,\gamma) &= 1-\bar{G}(x)+P(X=x) \\ &= 1-\left[\frac{\alpha\bar{F}(x+1)}{1-\bar{\alpha}\bar{F}(x+1)}\right]^{\gamma}; \quad x=0,1,2,... \end{aligned} \tag{2.4}$$

The hazard rate is given by

$$\begin{aligned} h(x;\alpha,\gamma) &= \frac{g(x)}{\bar{G}(x)} \\ &= 1-\left[\frac{\bar{F}(x+1)(1-\bar{\alpha}\bar{F}(x))}{\bar{F}(x)(1-\bar{\alpha}\bar{F}(x+1))}\right]^{\gamma}. \end{aligned} \tag{2.5}$$

The reversed hazard rate is

$$\begin{aligned} r^{*}(x;\alpha,\gamma) &= \frac{g(x)}{G(x)} \\ &= \frac{\left[\frac{\alpha\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}\right]^{\gamma}-\left[\frac{\alpha\bar{F}(x+1)}{1-\bar{\alpha}\bar{F}(x+1)}\right]^{\gamma}}{1-\left[\frac{\alpha\bar{F}(x+1)}{1-\bar{\alpha}\bar{F}(x+1)}\right]^{\gamma}}, \end{aligned}$$

and the second rate of failure is given by

$$\begin{aligned} r^{**}(x;\alpha,\gamma) &= \log\left[\frac{\bar{G}(x)}{\bar{G}(x+1)}\right] \\ &= \gamma\left[\frac{\bar{F}(x)(1-\bar{\alpha}\bar{F}(x+1))}{\bar{F}(x+1)(1-\bar{\alpha}\bar{F}(x))}\right]. \end{aligned}$$

2.2 Probability generating function, moments and quantiles

The probability generating function (pgf) of X is given by

$$P_X(s)=1+\alpha^{\gamma}(s-1)\sum_{x=1}^{\infty}s^{x-1}\left[\frac{\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}\right]^{\gamma}.$$

The recurrence relation of generating probabilities of discrete generalized Marshall-Olkin distribution is

$$g(x+1)=\frac{\left[\frac{\alpha\bar{F}(x+1)}{1-\bar{\alpha}\bar{F}(x+1)}\right]^{\gamma}-\left[\frac{\alpha\bar{F}(x+2)}{1-\bar{\alpha}\bar{F}(x+2)}\right]^{\gamma}}{\left[\frac{\alpha\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}\right]^{\gamma}-\left[\frac{\alpha\bar{F}(x+1)}{1-\bar{\alpha}\bar{F}(x+1)}\right]^{\gamma}}g(x); \quad x=0,1,2,...$$

In particular

$$g(0)=1-\left[\frac{\alpha\bar{F}(1)}{1-\bar{\alpha}\bar{F}(1)}\right]^{\gamma}.$$

The mean and variance of the random variable X are respectively

$$E(X) = \alpha^{\gamma} \sum_{x=1}^{\infty} \left[\frac{\bar{F}(x)}{1 - \bar{\alpha}\bar{F}(x)} \right]^{\gamma} \tag{2.6}$$

and

$$V(X) = \alpha^{\gamma} \sum_{x=1}^{\infty} (2x-1) \left[\frac{\bar{F}(x)}{1 - \bar{\alpha}\bar{F}(x)} \right]^{\gamma} - \left(\alpha^{\gamma} \sum_{x=1}^{\infty} \left[\frac{\bar{F}(x)}{1 - \bar{\alpha}\bar{F}(x)} \right]^{\gamma} \right)^{2}. \tag{2.7}$$

The quantiles x_q and median of the distribution are

$$x_q = F^{-1} \left[\frac{\alpha[1 - (1-u)^{\frac{1}{\gamma}}]}{\alpha + \bar{\alpha}(1-u)^{\frac{1}{\gamma}}} \right] - 1, \tag{2.8}$$

and

$$Median = F^{-1} \left[\frac{\alpha(2^{\frac{1}{\gamma}} - 1)}{\alpha 2^{\frac{1}{\gamma}} + \bar{\alpha}} \right] - 1. \tag{2.9}$$

Now, we study one member of this family of distributions in detail.

3 Discrete generalized Marshall-Olkin exponential distribution

The cdf and survival function of exponential distribution with scale parameter λ are $F(x;\lambda) = 1 - e^{-\lambda x}$ and $\bar{F}(x;\lambda) = e^{-\lambda x}$ respectively. Let $e^{-\lambda} = \theta$, $0 < \theta < 1$. Hence using (2.4), the cdf of the resulting discrete distribution is given by

$$G(x;\alpha,\gamma,\theta) = 1 - \left[\frac{\alpha\theta^{x+1}}{1 - \bar{\alpha}\theta^{x+1}} \right]^{\gamma}; \quad x = 0, 1, 2, ... \tag{3.1}$$

We call the random variable X with cdf (3.1) as discrete generalized Marshall-Olkin exponential (DGMOE) distribution with parameters $\alpha > 0$, $0 < \theta < 1$, $\gamma > 0$ and $\bar{\alpha} = 1 - \alpha$ and denote it by DGMOE(α, γ, θ).

The pmf of DGMOE(α, γ, θ) is given by

$$g(x) = \left[\frac{\alpha\theta^{x}}{1 - \bar{\alpha}\theta^{x}} \right]^{\gamma} - \left[\frac{\alpha\theta^{x+1}}{1 - \bar{\alpha}\theta^{x+1}} \right]^{\gamma}; \quad x = 0, 1, 2, ... \tag{3.2}$$

In Figure 1, plots of the pmf of DGMOE(α, γ, θ) distribution are presented for some parameter values.

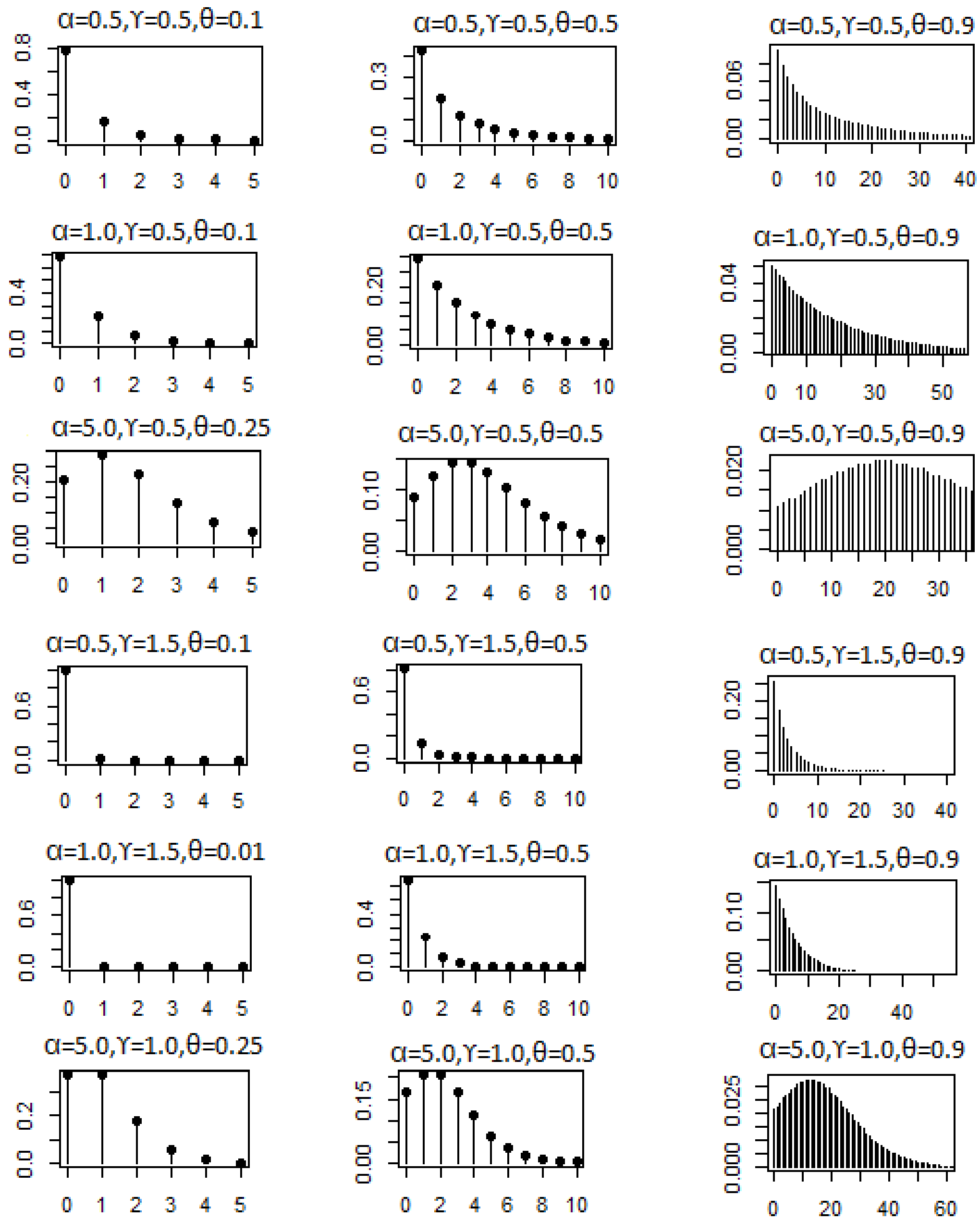

Figure 1. Pmf of DGMOE(α, γ, θ) distribution for some parameter values.

Some discrete distributions that are special cases of DGMOE distribution are:

1. When $\gamma = 1$,

$$g(x;\alpha,1,\theta) = \frac{\alpha(1-\theta)\theta^x}{(1-\bar{\alpha}\theta^{x+1})(1-\bar{\alpha}\theta^x)}; \quad x = 0,1,2,...$$

   which is the generalized geometric distribution introduced and studied by Gómez-Déniz (2010).

2. When $\alpha = 1$,

$$g(x;1,\gamma,\theta) = (\theta^x)^\gamma - (\theta^{x+1})^\gamma = (1-\theta^\gamma)\theta^{\gamma x}; \quad x = 0,1,2,...$$

   which is the geometric distribution with probability of success as $(1-\theta^\gamma)$.

The survival function of DGMOE(α,γ,θ) distribution is

$$\bar{G}(x;\alpha,\gamma,\theta) = \left[\frac{\alpha\theta^x}{1-\bar{\alpha}\theta^x}\right]^\gamma. \tag{3.3}$$

The hazard rate is given by

$$h(x;\alpha,\gamma,\theta) = 1 - \left[\frac{\theta(1-\bar{\alpha}\theta^x)}{1-\bar{\alpha}\theta^{x+1}}\right]^\gamma. \tag{3.4}$$

As we see from Figure 2, the hazard rate function of the new distribution can be decreasing, constant or increasing depending on its parameters values, and hence presents a very flexible behavior.

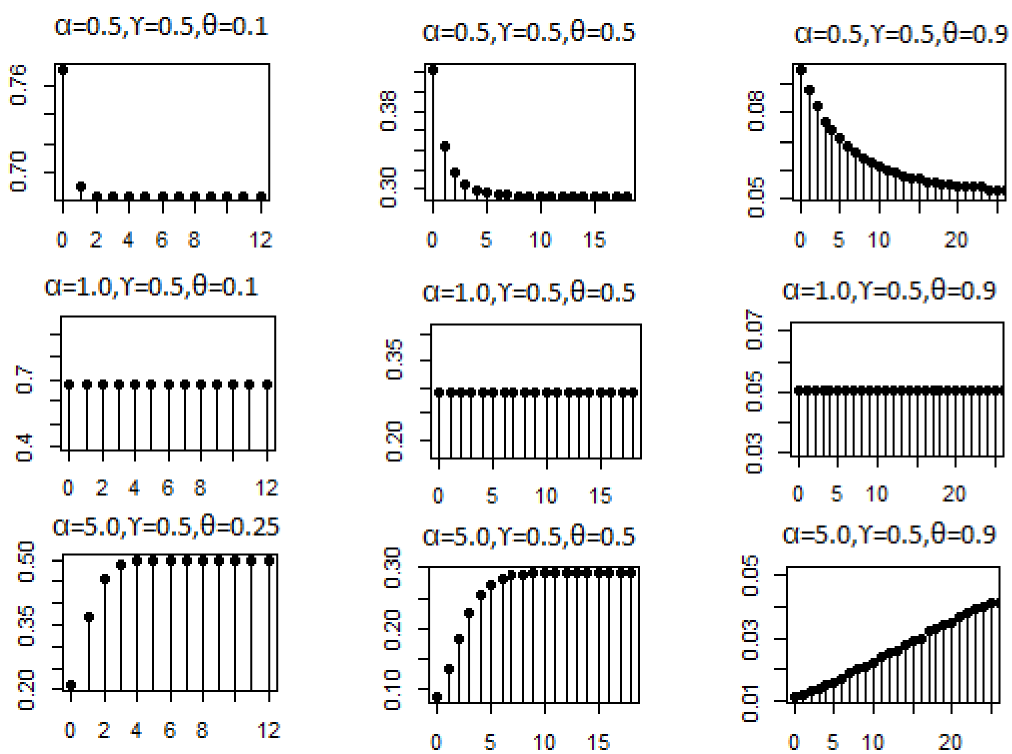

Figure 2. Hazard rate function of the DGMOE(α, γ, θ) distribution for some parameter values.

Also the reverse hazard rate and second rate of failure are respectively,

$$r^*(x;\alpha,\gamma,\theta) = \frac{\left[\frac{\alpha\theta^x}{1-\bar{\alpha}\theta^x}\right]^\gamma - \left[\frac{\alpha\theta^{x+1}}{1-\bar{\alpha}\theta^{x+1}}\right]^\gamma}{1 - \left[\frac{\alpha\theta^{x+1}}{1-\bar{\alpha}\theta^{x+1}}\right]^\gamma},$$

and

$$r^{**}(x;\alpha,\gamma,\theta) = \gamma \left[\frac{\theta^x(1-\bar{\alpha}\theta^{x+1})}{\theta^{x+1}(1-\bar{\alpha}\theta^x)}\right].$$

Now let $k > 0$ be integer. If $|z| < 1$, we have the binomial series representation

$$(1-z)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k+j)}{\Gamma(k)j!} z^j.$$

Hence equation (3.2), for $0 < \alpha < 1$, can be written as

$$\begin{aligned} g(x;\alpha,\gamma,\theta) &= \sum_{j=0}^{\infty} \frac{\Gamma(\gamma+j)}{\Gamma(\gamma)j!}\alpha^{\gamma}\bar{\alpha}^{j}\theta^{(\gamma+j)x} - \sum_{j=0}^{\infty} \frac{\Gamma(\gamma+j)}{\Gamma(\gamma)j!}\alpha^{\gamma}\bar{\alpha}^{j}\theta^{(\gamma+j)(x+1)} \\ &= \sum_{j=0}^{\infty} \frac{\Gamma(\gamma+j)}{\Gamma(\gamma)j!}\alpha^{\gamma}\bar{\alpha}^{j}\theta^{(\gamma+j)x}(1-\theta^{(\gamma+j)}) \\ &= \sum_{j=0}^{\infty} u_j(\alpha,\gamma)(1-\theta^{(\gamma+j)})\theta^{(\gamma+j)x}, \quad x = 0,1,2,\ldots, \end{aligned} \tag{3.5}$$

where

$$u_j(\alpha,\gamma) = \frac{\Gamma(\gamma+j)}{\Gamma(\gamma)j!}\alpha^{\gamma}\bar{\alpha}^{j}.$$

Since, the above pmf (3.5) is a linear combination of the geometric distribution with probability of success as $(1-\theta^{(\gamma+j)})$, we can establish several properties of the DGMOE(α,γ,θ) distribution using geometric distribution.

So the moment and probability generating functions of DGMOE(α,γ,θ) distribution are respectively, given by

$$M_X(t) = \sum_{j=0}^{\infty} u_j(\alpha,\gamma)\frac{1-\theta^{\gamma+j}}{1-\theta^{\gamma+j}e^t}$$

and

$$P_X(s) = \sum_{j=0}^{\infty} u_j(\alpha,\gamma)\frac{1-\theta^{\gamma+j}}{1-\theta^{\gamma+j}s}$$

The factorial moments are given by

$$E[X(X-1)(X-2)...(X-r+1)] = \sum_{j=o}^{\infty} u_j(\alpha,\gamma)\left(\frac{\theta^{\gamma+j}}{1-\theta^{\gamma+j}}\right)^r$$

for $r = 1, 2, \ldots$. Hence, the mean and variance of the DGMOE(α,γ,θ) distribution can be obtained as

$$E(X) = \sum_{j=o}^{\infty} u_j(\alpha,\gamma)\left(\frac{\theta^{\gamma+j}}{1-\theta^{\gamma+j}}\right), \tag{3.6}$$

and

$$V(X) = \sum_{j=o}^{\infty} u_j(\alpha,\gamma)\left(\frac{\theta^{\gamma+j}}{1-\theta^{\gamma+j}}\right)\left(1+\frac{\theta^{\gamma+j}}{1-\theta^{\gamma+j}}\right) - \left\{\sum_{j=o}^{\infty} u_j(\alpha,\gamma)\left(\frac{\theta^{\gamma+j}}{1-\theta^{\gamma+j}}\right)\right\}^2. \tag{3.7}$$

In addition, the median of the DGMOE(α, γ, θ) distribution is given by

$$Median = \left[\frac{1}{\log \theta} \left\{ \log \frac{(1-(1/2))^{\frac{1}{\gamma}}}{1+\bar{\alpha}(1-(1/2))^{\frac{1}{\gamma}}} \right\} - 1 \right],$$

where [.] denotes the integer part of the expression.

The recurrence relation for generating probabilities of DGMOE(α, γ, θ) is given by

$$g(x+1) = \frac{\left[\{\theta(1-\bar{\alpha}\theta^{x+2})\}^{\gamma} - \{\theta^2(1-\bar{\alpha}\theta^{x+1})\}^{\gamma}\right](1-\bar{\alpha}\theta^{x})^{\gamma}}{\left[(1-\bar{\alpha}\theta^{x+1})^{\gamma} - \{\theta(1-\bar{\alpha}\theta^{x})\}^{\gamma}\right](1-\bar{\alpha}\theta^{x+2})^{\gamma}} g(x); \quad x = 0, 1, 2, \dots,$$

with

$$g(0) = 1 - \left[\frac{\alpha\theta}{1-\bar{\alpha}\theta} \right]^{\gamma}.$$

The mean and variance of the DGMOE(α, γ, θ) distribution are calculated in Table 1 for different values of its parameters. The mean and variance increases, when α and θ increases, and decreases when γ increases. Moreover, depending on the values of the parameters, the mean of the distribution is smaller than its variance.

$\gamma = 0.5$					
α/θ	0.1	0.3	0.5	0.7	0.9
0.1	0.1512(0.2639)	0.4173(1.1823)	0.8733(3.8925)	1.9483(15.4421)	7.4451(181.3613)
0.5	0.3330(0.5250)	0.8950(2.1821)	1.8214(6.8589)	3.9333(26.3482)	14.4175(303.7673)
1.0	0.4625(0.6764)	1.2110(2.6776)	2.4142(8.2426)	5.1222(31.3591)	18.4868(360.2498)
2.0	0.6325(0.8359)	1.5958(3.1453)	3.1110(9.5370)	6.4967(36.0776)	23.1629(413.6707)
$\gamma = 1.0$					
α/θ	0.1	0.3	0.5	0.7	0.9
0.1	0.0121(0.0145)	0.0548(0.0904)	0.1502(0.3518)	0.4188(1.5661)	2.0096(19.8165)
0.5	0.0582(0.0672)	0.2431(0.3728)	0.6070(1.2999)	1.5013(5.2372)	6.0963(61.4726)
1.0	0.1111(0.1235)	0.4286(0.6122)	1.0000(2.0000)	2.3333(7.7778)	9.0000(90.0000)
2.0	0.2038(0.2113)	0.7023(0.9075)	1.5290(2.8139)	3.4016(10.7212)	12.6620(123.2068)
$\gamma = 2.0$					
α/θ	0.1	0.3	0.5	0.7	0.9
0.1	0.0001(0.0001)	0.0018(0.0020)	0.0096(0.0127)	0.0476(0.0831)	0.4369(1.4695)
0.5	0.0028(0.0028)	0.0336(0.0377)	0.1373(0.1862)	0.4726(0.8985)	2.4473(11.3801)
1.0	0.0101(0.1020)	0.0989(0.1087)	0.3333(0.4444)	0.9608(1.8839)	4.2632(22.4377)
2.0	0.0335(0.3313)	0.2433(0.2515)	0.6726(0.8637)	1.6958(3.3943)	6.8415(39.3809)

Table 1. Mean(Variance) of DGMOE(α, γ, θ) for different values of parameters.

3.1 Infinite divisibility

According to Steutel and van Harn (2004, pp.56), if $g(x)$, $x \in \mathbf{N}_0$ is infinitely divisible, then $g(x) < e^{-1}$ for all $x \in \mathbf{N}$. However, e.g. in DGMOE (6.0,1.5,0.25) distribution, we can see that $g(1) = 0.392 > e^{-1}$=0.367. Therefore, in general, DGMOE (α, γ, θ) distribution is not infinitely divisible. In addition, since the class of self decomposable and stable distributions, in their discrete concept, are subclass of infinitely divisible distributions, we can conclude that DGMOE distribution can be neither self decomposable nor stable, in general.

4 Maximum likelihood estimation

4.1 Estimation

We now consider method of maximum likelihood for estimating the parameter vector $\lambda = (\alpha, \gamma, \theta)^T$ of DGMOE distribution. Assume that $x = (x_1, x_2 ..., x_n)^T$ is a random sample of size n from DGMOE(α, γ, θ) distribution. Then the log-likelihood function is given by

$$\log L(x; \alpha, \gamma, \theta) = \sum_{i=1}^{n} \log \left(\left[\frac{\alpha \theta^{x_i}}{1 - \bar{\alpha}\theta^{x_i}} \right]^{\gamma} - \left[\frac{\alpha \theta^{x_{i+1}}}{1 - \bar{\alpha}\theta^{x_{i+1}}} \right]^{\gamma} \right).$$

Differentiating the log-likelihood function with respect to the parameters, we obtain

$$\begin{aligned}
\frac{\partial \log L}{\partial \alpha} &= \sum_{i=1}^{n} \frac{\left(\frac{\alpha\theta^{x_i}}{1-\bar{\alpha}\theta^{x_i}}\right)^{\gamma} \frac{\gamma(1-\theta^{x_i})}{\alpha(1-\bar{\alpha}\theta^{x_i})} - \left(\frac{\alpha\theta^{x_i+1}}{1-\bar{\alpha}\theta^{x_i+1}}\right)^{\gamma} \frac{\gamma(1-\theta^{x_i+1})}{\alpha(1-\bar{\alpha}\theta^{x_i+1})}}{\left(\frac{\alpha\theta^{x_i}}{1-\bar{\alpha}\theta^{x_i}}\right)^{\gamma} - \left(\frac{\alpha\theta^{x_i+1}}{1-\bar{\alpha}\theta^{x_i+1}}\right)^{\gamma}}, \\
\frac{\partial \log L}{\partial \gamma} &= \sum_{i=1}^{n} \frac{\left(\frac{\alpha\theta^{x_i}}{1-\bar{\alpha}\theta^{x_i}}\right)^{\gamma} \log\left(\frac{\alpha\theta^{x_i}}{1-\bar{\alpha}\theta^{x_i}}\right) - \left(\frac{\alpha\theta^{x_i+1}}{1-\bar{\alpha}\theta^{x_i+1}}\right)^{\gamma} \log\left(\frac{\alpha\theta^{x_i+1}}{1-\bar{\alpha}\theta^{x_i+1}}\right)}{\left(\frac{\alpha\theta^{x_i}}{1-\bar{\alpha}\theta^{x_i}}\right)^{\gamma} - \left(\frac{\alpha\theta^{x_i}}{1-\bar{\alpha}\theta^{x_i}}\right)^{\gamma}}, \\
\frac{\partial \log L}{\partial \theta} &= \sum_{i=1}^{n} \frac{\gamma x_i \theta(1-\bar{\alpha}\theta^{x_i}) \left(\frac{\alpha\theta^{x_i}}{1-\bar{\alpha}\theta^{x_i}}\right)^{\gamma} - \frac{\gamma(x_i+1)}{\theta(1-\bar{\alpha}\theta^{x_i+1})} \left(\frac{\alpha\theta^{x_i+1}}{1-\bar{\alpha}\theta^{x_i+1}}\right)^{\gamma}}{\left(\frac{\alpha\theta^{x_i}}{1-\bar{\alpha}\theta^{x_i}}\right)^{\gamma} - \left(\frac{\alpha\theta^{x_i}}{1-\bar{\alpha}\theta^{x_i}}\right)^{\gamma}}.
\end{aligned}$$

The maximum likelihood estimates (MLEs) can be obtained numerically solving the equation $\frac{\partial \log L}{\partial \alpha} = 0, \frac{\partial \log L}{\partial \gamma} = 0$ and $\frac{\partial \log L}{\partial \theta} = 0$. Let the estimates be $\hat{\lambda} = (\hat{\alpha}, \hat{\gamma}, \hat{\theta})^T$ which can be obtained by a numerical method such as the three variable Newton-Raphson type procedure.

For interval estimation and hypothesis tests on the model parameters, we require the information matrix. The 3 x 3 observed information matrix is

$$I_n(\hat{\lambda}) = - \begin{bmatrix} \frac{\partial^2 \log L}{\partial \alpha^2} & \frac{\partial^2 \log L}{\partial \alpha \partial \gamma} & \frac{\partial^2 \log L}{\partial \alpha \partial \theta} \\ \frac{\partial^2 \log L}{\partial \gamma \partial \alpha} & \frac{\partial^2 \log L}{\partial \gamma^2} & \frac{\partial^2 \log L}{\partial \gamma \partial \theta} \\ \frac{\partial^2 \log L}{\partial \theta \partial \alpha} & \frac{\partial^2 \log L}{\partial \theta \partial \gamma} & \frac{\partial^2 \log L}{\partial \theta^2} \end{bmatrix}$$

whose elements are obtained by differentiating partially, the first partial derivative of the likelihood function with respect to the parameters. We can easily show that the DGMOE family satisfies the regularity conditions which are fulfilled for parameters in the interior parameter space but not on the boundary. Hence the MLE vector $\hat{\lambda}$ is consistent and asymptotically normal. That is, $I_n^{\frac{1}{2}}(\hat{\lambda} - \lambda)$ converges in distribution to multivariate normal with mean vector as zero vector and covariance matrix as identity matrix.

We can use the normal distribution of $\hat{\lambda}$ to construct approximate confidence regions for the parameters. The asymptotic $100(1-\eta)\%$ confidence interval for each parameter λ_i is given by

$$(\hat{\lambda}_i - Z_{\frac{\eta}{2}}\sqrt{\hat{J}_{ii}}\ \hat{\lambda}_i,\ \hat{\lambda}_i + Z_{\frac{\eta}{2}}\sqrt{\hat{J}_{ii}}\ \hat{\lambda}_i), \quad i = 1, 2, 3\,,$$

where $\hat{J}_{ii}$ denotes the $(i,i)^{th}$ diagonal elements of $I_n^{-1}(\hat{\lambda})$ and $Z_{\frac{\eta}{2}}$ is the $(1-\frac{\eta}{2})^{th}$ quantile of the standard normal distribution.

4.2 Simulation

Let X be a random variable which follow generalized Marshall-Olkin exponential distribution with cdf

$$F(x;\alpha,\gamma,\lambda) = 1 - \left(\frac{\alpha e^{-\lambda x}}{1-\bar{\alpha}e^{-\lambda x}}\right)^{\gamma}; \quad \alpha > 0,\, \gamma > 0,\, \lambda > 0.$$

Then $[X]$ has the DGMOE(α,γ,θ) distribution in which $0 < \theta = e^{-\lambda} < 1$. Therefore, we can simulate DGMOE(α,γ,θ) random variable from the corresponding continuous generalized Marshall-Olkin exponential distribution. The maximum likelihood estimate of $(\alpha,\gamma,\theta)^T$ from DGMOE(α,γ,θ) distribution along with their standard error for different sample sizes n based on simulation is given in Table 2. Standard errors are obtained by means of the asymptotic covariance matrix of the MLEs of DGMOE(α,γ,θ) parameters when the Newton-Raphson method converges using **R** software. It can be seen that the deviation between actual value and estimated value is relatively small. Also when the sample size increases, the standard deviation of the estimators decreases.

	α=0.5	γ=0.5	θ=0.3	α=1.5	γ=0.5	θ=0.5
n	$\hat{\alpha}(\hat{SE}(\hat{\alpha}))$	$\hat{\gamma}(\hat{SE}(\hat{\gamma}))$	$\hat{\theta}(\hat{SE}(\hat{\theta}))$	$\hat{\alpha}(\hat{SE}(\hat{\alpha}))$	$\hat{\gamma}(\hat{SE}(\hat{\gamma}))$	$\hat{\theta}(\hat{SE}(\hat{\theta}))$
50	0.536(2.632)	0.549(0.447)	0.372(0.321)	1.847(1.847)	0.624(0.442)	0.656(0.656)
100	0.498(1.457)	0.493(0.329)	0.265(0.138)	1.486(1.364)	0.568(0.369)	0.439(0.524)
200	0.506(1.099)	0.515(0.197)	0.326(0.095)	1.594(1.140)	0.424(0.245)	0.487(0.396)
500	0.490(1.003)	0.533(0.121)	0.309(0.059)	1.512(0.842)	0.478(0.214)	0.526(0.215)
	α=0.5	γ=1.0	θ=0.3	α=1.5	γ=1.0	θ=0.5
n	$\hat{\alpha}(\hat{SE}(\hat{\alpha}))$	$\hat{\gamma}(\hat{SE}(\hat{\gamma}))$	$\hat{\theta}(\hat{SE}(\hat{\theta}))$	$\hat{\alpha}(\hat{SE}(\hat{\alpha}))$	$\hat{\gamma}(\hat{SE}(\hat{\gamma}))$	$\hat{\theta}(\hat{SE}(\hat{\theta}))$
50	0.548(1.884)	0.914(1.023)	0.463(0.816)	1.764(3.689)	0.975(1.326)	0.523(0.264)
100	0.478(1.623)	1.236(0.894)	0.383(0.743)	1.543(2.831)	0.994(1.216)	0.518(0.203)
200	0.438(0.947)	1.112(0.786)	0.312(0.701)	1.326(1.989)	1.136(0.978)	0.489(0.181)
500	0.534(0.823)	0.895(0.780)	0.263(0.631)	1.563(1.532)	0.990(0.723)	0.474(0.132)
	α=0.5	γ=1.5	θ=0.3	α=1.5	γ=1.5	θ=0.7
n	$\hat{\alpha}(\hat{SE}(\hat{\alpha}))$	$\hat{\gamma}(\hat{SE}(\hat{\gamma}))$	$\hat{\theta}(\hat{SE}(\hat{\theta}))$	$\hat{\alpha}(\hat{SE}(\hat{\alpha}))$	$\hat{\gamma}(\hat{SE}(\hat{\gamma}))$	$\hat{\theta}(\hat{SE}(\hat{\theta}))$
50	0.589(1.473)	1.327(1.687)	0.354(0.324)	1.548(2.140)	1.527(3.051)	0.751(0.678)
100	0.506(1.212)	1.746(1.326)	0.333(0.235)	1.494(1.681)	1.549(2.847)	0.716(0.436)
200	0.514(1.016)	1.658(1.089)	0.279(0.218)	1.587(1.439)	1.498(2.181)	0.688(0.326)
500	0.512(0.830)	1.439(0.985)	0.320(0.191)	1.465(1.235)	1.506(1.550)	0.708(0.223)

Table 2: MLEs of the DGMOE parameters for different n.

5 Application

In this section, we illustrate the flexibility of the proposed distribution using a real data set. This data is taken from Lawless (2003). The data set gives the number of cycles to failure for twenty five 100 cm specimens of yarn, tested at a particular strain level. Maximum likelihood estimation is used to obtain the parameter estimates of the models (using **R** software). We compare the fit of the DGMOE distribution with the following discrete life time distributions:

(a). Geometric (G) distribution having pmf

$$P(X = x; q) = (1 - q)q^x; \quad 0 < q < 1,\, x = 0,\, 1,\, 2, \ldots .$$

(b). Exponentiated geometric (EG) distribution (Chakraborty and Gupta (2015)) having pmf

$$P(X = x; q, \alpha) = (1-q^{x+1})^{\alpha}-(1-q^x)^{\alpha}; \quad 0 < q < 1,\, \alpha > 0,\, \bar{\alpha} = 1-\alpha,\, x = 0,\, 1,\, 2, \ldots .$$

(c). Generalized geometric (GG) distribution (Gómez-Déniz (2010)) having pmf

$$P(X = x; \alpha, q) = \left[\frac{1 - q^{x+1}}{1 - \bar{\alpha}q^{x+1}}\right]-\left[\frac{1 - q^x}{1 - \bar{\alpha}q^x}\right]; \, 0 < q < 1, \alpha > 0, \bar{\alpha} = 1-\alpha, x = 0, 1, 2, \ldots .$$

(d). Kumaraswamy-geometric (KGD) distribution (Akinsete et al. (2014)) having pmf

$$\begin{aligned} P(X = x; \alpha, \beta, q) &= [1 - (1 - q^x)^{\alpha}]^{\beta} - [1 - (1 - q^{x+1})^{\alpha})]^{\beta}; \\ &\quad 0 < q < 1,\, \alpha > 0,\, \beta > 0,\, \bar{\alpha} = 1 - \alpha,\, x = 0, 1, 2, \ldots . \end{aligned}$$

and

(e). Exponentiated generalized geometric (EGG) distribution (Bidram et al. (2016)) having pmf

$$\begin{aligned} P(X = x; \alpha, \beta, q) &= \left[\frac{1 - q^{x+1}}{1 - \bar{\alpha}q^{x+1}}\right]^{\beta} - \left[\frac{1 - q^x}{1 - \bar{\alpha}q^x}\right]^{\beta}; \\ &\quad 0 < q < 1,\, \alpha > 0,\, \beta > 0,\, \bar{\alpha} = 1 - \alpha,\, x = 0, 1, 2, \ldots . \end{aligned}$$

The values of the log-likelihood function (-log L), AIC (Akaike Information Criterion), $AICc$ (Akaike Information Criterion with correction),BIC (Bayesian Information Criterion) and $HQIC$ (Hannon-Quinn Information Criterion) are calculated for the distributions in order to verify which distribution fits better to the data. The better distribution corresponds to smaller -log L, AIC, $AICc$, BIC, and $HQIC$. Here, $AIC = -2\,\log L + 2k$, $AICc = -2\,\log\, L + \frac{2kn}{n-k-1}$, $BIC = -2\,\log L + k\,\log n$, and $HQIC = -2\,\log L + 2k\log(\log(n))$, where L is the likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters and n is the sample size.

The data set is given below:

15 20 38 42 61 76 86 98 121 146 149 157 175 176 180 180 198 220 224 251 264 282 321 325 653.

The MLE of parameters of the models and the measures $-\log L$, AIC, $AICc$, BIC and $HQIC$ are given in Table 3.

	Estimates	$-\log L$	AIC	$AICc$	BIC	$HQIC$
G	$q = 0.9944$	154.6595	311.3190	311.4619	312.7202	311.7673
EG	$q = 0.9919$, $\alpha = 1.8877$	152.4965	308.9902	309.4346	311.7926	309.8867
GG	$q = 0.9888$, $\alpha = 4.9262$	152.2990	308.5978	309.0422	311.4002	309.4943
KGD	$\alpha = 0.1000$, $\beta = 0.0999$ $q = 0.9499$	157.7331	321.4662	322.3893	325.6698	322.8110
EGG	$\alpha = 2.9559$, $\beta = 0.9896$ $q = 1.2726$	152.2170	310.4348	311.3579	314.6384	311.7796
DGMOE	$\alpha = 0.2596$, $\gamma = 0.0047$ $\theta = 0.3200$	149.6051	305.2102	306.1333	309.4138	306.5550

Table 3. Parameter estimates and goodness of fit for various models fitted to the yarn data.

From the **Table 3**, we can see that $-\log L$, AIC, $AICc$, BIC and $HQIC$ are smallest for DGMOE with $-\log L$ =149.6051, AIC = 305.2102, $AICc$=306.1333, BIC = 309.4138 and $HQIC$ = 306.5550. Hence DGMOE model gives a satisfactory fit to this data set.

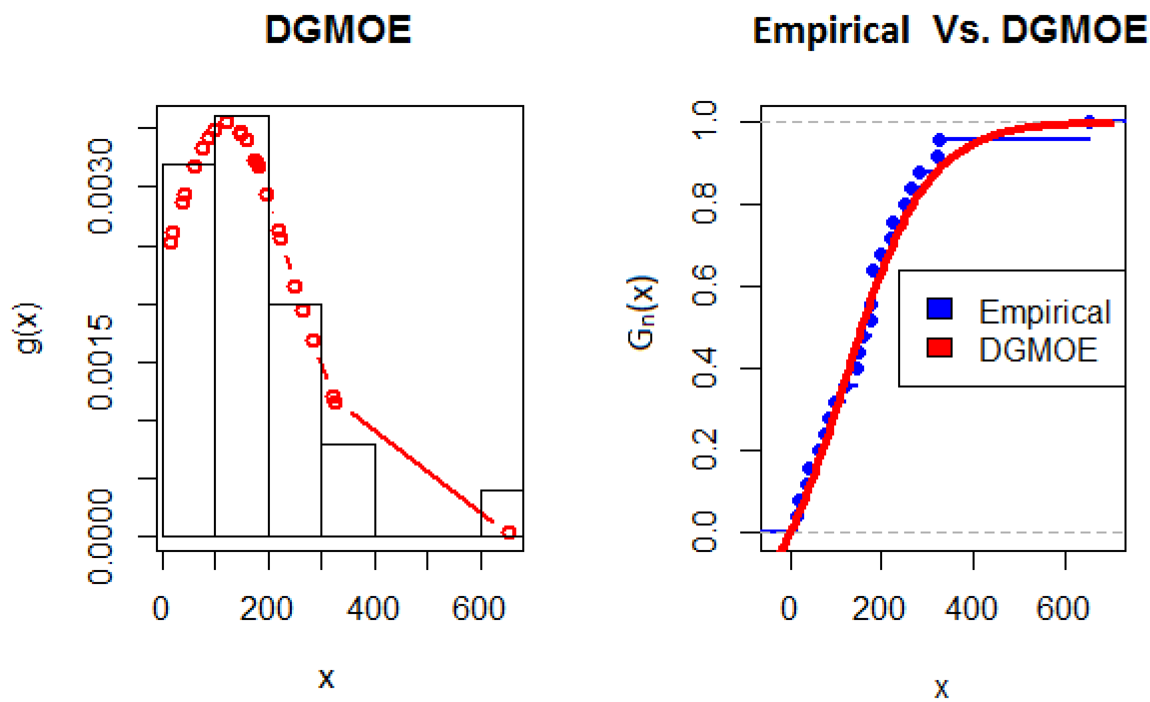

Figure 3: Plots of the estimated pmf and cdf of the DGMOE model for the yarn data set.

To test the null hypothesis H_0: GG versus the alternative hypothesis H_1: DGMOE, or equivalently $H_0 : \gamma = 1$ versus $H_1 : \gamma \neq 1$, we use the likelihood ratio statistic whose value is 5.3876 (p-value =0.0203). As a result, the null model GG, is rejected in favor of the alternative model DGMOE at any level > 0.0203.

6 Conclusion

In this paper, we have introduced a new distribution, discrete generalized Marshall-Olkin exponential (DGMOE) belonging to the resilience parameter family. This new distribution contains generalized geometric distribution of Gómez-Dénniz (2010) and geometric distribution. Moreover, the DGMOE distribution coincides with the discrete counterpart of the generalized Marshall-Olkin exponential distribution. We have studied some basic statistical and mathematical properties of the new model and illustrated that the hazard rate function of the new model can be increasing or decreasing. Also the distribution can be positively or negatively skewed, and leptokurtic. From the application presented here, it can be seen that the DGMOE distribution appears to be more suitable for modeling real data set and is a better alternative to some existing distributions.

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Mixtures of Gamma Distributions and Harris Infinite Divisibility

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ABSTRACT

We look at the relation between mixtures of gamma distributions and Harris infinite divisibility. We show that mixtures of gamma distributions with shape parameter $\frac{1}{k}, k > 1$ integer, are Harris infinitely divisible. We also prove that Harris infinitely divisible distributions are geometrically infinitely divisible and that all Harris infinitely divisible laws are not mixtures of gamma distributions.

Key words and Phrases: *mixtures of gamma, completely monotone, geometric infinite divisibility, Harris infinite divisibility.*

1 Introduction

From Pillai and Sandhya (1990) we know that a distribution function ($d.f$) has complete monotone derivative (CMD) *iff* it is a mixture of exponential distributions and further they formed a proper subclass of the class of geometrically infinitely divisible (GID) distributions. A curiosity is whether these results can be extended to another class of mixtures and a corresponding notion of random ($\mathcal{N}$) infinite

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divisibility where $\mathcal{N}$ has a specified discrete distribution. An important hint in this direction is: exponential distribution is invariant under a random-sum ($\mathcal{N}$-sum) if $\mathcal{N}$ is geometric with mean $\frac{1}{p}, 0 < p < 1$.

Satheesh *et al.* (2002) proved that the gamma distribution with Laplace Transform (LT) $\left(\frac{1}{1+\lambda}\right)^{\alpha}, \alpha > 0$ is invariant under an $\mathcal{N}$-sum *iff* $\alpha = \frac{1}{k}$ and the *r.v* $\mathcal{N}$ has a Harris (m, k) distribution with probability generating function (PGF)

$$P(s) = \left(\frac{s^k}{m-(m-1)s^k}\right)^{\frac{1}{k}}, k \geq 1 \text{integer}. \tag{1.1}$$

Further, since the LT $\phi(\lambda) = \left(\frac{1}{1+\lambda}\right)^{\frac{1}{k}}$ satisfies $\phi\{m\phi^{-1}(s)\} = \left(\frac{s^k}{m-(m-1)s^k}\right)^{\frac{1}{k}}$, Harris distribution qualifies for $\mathcal{N}$ in the definition of $\mathcal{N}$-infinite divisibility of Gnedenko and Korolev (1996, p.144) and Harris infinite divisibility (HID) was introduced in Satheesh *et al.* (2008).

Harris(m, k) distribution was introduced in Harris (1948) and was studied in some detail by Sherly (2008), Sandhya *et al.* (2008) and Lovely (2015). The mean of Harris distribution is $m = \frac{1}{p}, 0 < p < 1$ and for $k = 1$, it becomes the geometric distribution with mean $\frac{1}{p}$. Hence we discuss the case $k > 1$ here.

Definition 1.1. *A r.v Y is HID if for every $p \in (0, 1)$,*

$$Y = X_1^{(p)} + \ldots + X_{H_p}^{(p)} \tag{1.2}$$

where $X_1^{(p)}, X_2^{(p)}, \ldots$ are i.i.d r.vs independent of H_p for each $p \in (0, 1)$ and H_p has a Harris (m, k) distribution with mean $m = 1/p$ and PGF (1.1).

Putting $p = 1/m$ and taking characteristic functions (CF) on both sides of (1.2) we get; A CF $h(t)$ is HID if for every $p \in (0, 1)$ and for a fixed integer $k > 1$ there exists another CF $\phi_p(t)$ such that

$$h(t) = \left(\frac{p\phi_p^k(t)}{1-(1-p)\phi_p^k(t)}\right)^{\frac{1}{k}}.$$

This representation also suggests its close relation with the notion of GID. Consequently the CF of a HID distribution has the form, Satheesh *et al.* (2008),

$$h(t) = \left(\frac{1}{1+k\psi(t)}\right)^{\frac{1}{k}}, \tag{1.3}$$

where $e^{-\psi(t)}$ is infinitely divisible (ID) or $\frac{1}{1+\psi(t)}$ is GID. For this reason the $'k'$ in $k\psi(t)$ does not have any role in the discussion here and hence we will drop this $'k'$ in the sequel. The LT of a non-negative HID $r.v$ is $\left(\frac{1}{1+\psi(\lambda)}\right)^{\frac{1}{k}}$, such that the LT $e^{-\psi(\lambda)}$ is ID or $\frac{1}{1+\psi(\lambda)}$ is GID or $\psi(\lambda)$ has CMD and $\psi(0)=0$. We record this in the following theorem.

Theorem 1.1. *A CF $h(t)$ is HID iff for an integer $k>1, h^k(t)=g(t)$, is a CF that is GID. A LT $h(\lambda)$ is HID iff $h^k(\lambda)=g(\lambda)$ is a LT that is GID.*

Now a corollary to property 4.6.2 in Gnedenko and Korolev (1996, p.145) is as given below.

Corollary 1.1. *The limit of a sequence of HID distributions is HID.*

In this note by gamma(α,β) we mean the gamma distribution with LT $\left(\frac{\beta}{\beta+\lambda}\right)^{\alpha}$ and in a mixture of gamma(α,β) we treat β as a $r.v.$ Here we extend lemma 2.2, theorem 2.1 and 2.2 of Pillai and Sandhya (1990). We show that mixtures of gamma$(\frac{1}{k},\beta)$ distributions is a proper subclass of the class of HID distributions and the class of HID distributions is a proper subclass of the class of GID distributions. Every distribution considered here has non-negative support.

2 Results.

We begin with a lemma.

Lemma 2.1. *A finite mixture of gamma$(\frac{1}{k},a_j)$ distributions, $k>1$ integer, is HID.*

Proof. Let $\phi(\lambda)$ be the LT of a finite mixture of gamma$(\frac{1}{k},a_j)$ distributions. Then,

$$\phi(\lambda)=\sum_{j=1}^{n} p_j\left(\frac{a_j}{a_j+\lambda}\right)^{\frac{1}{k}}.$$

Consider $\phi^k(\lambda)=\left\{\sum_{j=1}^{n} p_j\left(\frac{a_j}{a_j+\lambda}\right)^{\frac{1}{k}}\right\}^k$. By the proof of corollary 2.12.2 of Steutel (1970, p.51), $\phi^k(\lambda)$ is the LT of a mixture of exponentials. Hence by Pillai and Sandhya (1990) $\phi^k(\lambda)$ is GID and consequently $\phi(\lambda)$ is HID. $\square$

Theorem 2.1. *Let $\phi(\lambda)$ be the LT of a mixture of gamma$(\frac{1}{k},\beta)$ distributions, $k>1$ integer. Then $\phi(\lambda)$ is HID.*

Proof. $\phi(\lambda)=\int_0^\infty\left(\frac{\beta}{\beta+\lambda}\right)^{\frac{1}{k}}dG(\beta)$, where $G(\beta)$ is a *d.f.* Choose $\{p_j\}$ in lemma 2.1 such that $G(.)$ is its limit. Now the limit of a sequence of mixtures of gamma distributions is a mixture of gamma (Steutel and van Harn, 2004, p.344) and the limit of a sequence of HID distributions is again HID (corollary 1.1). Hence mixtures of gamma$(\frac{1}{k},\beta)$ distributions is HID. $\square$

This result also helps in deriving HID laws as illustrated below.

Example 2.1 Consider the gamma$(\frac{1}{k},\beta)$ distribution and let β has an exponential distribution with parameter 1. Then the mixture has $p.d.f$

$$\begin{aligned}\int_0^\infty\frac{\beta^{\frac{1}{k}}}{\Gamma\left(\frac{1}{k}\right)}e^{-\beta x}x^{\frac{1}{k}-1}e^{-\beta}d\beta &= \frac{x^{\frac{1}{k}-1}}{\Gamma\left(\frac{1}{k}\right)}\int_0^\infty\beta^{\frac{1}{k}}e^{-\beta(1+x)}d\beta\\ &= \frac{x^{\frac{1}{k}-1}}{\Gamma\left(\frac{1}{k}\right)}\int_0^\infty e^{-\beta(1+x)}\beta^{(\frac{1}{k}+1)-1}d\beta.\\ &= \frac{x^{\frac{1}{k}-1}}{\Gamma\left(\frac{1}{k}\right)}\frac{\Gamma\left(\frac{1}{k}+1\right)}{(1+x)^{\frac{1}{k}+1}}\\ &= \frac{\Gamma\left(\frac{1}{k}+1\right)}{\Gamma(\frac{1}{k})}\frac{x^{\frac{1}{k}-1}}{(1+x)^{\frac{1}{k}+1}},x>0,\end{aligned}$$

which is beta$(\frac{1}{k},1)$ distribution of the second kind and being a mixture of gamma distributions, is HID.

We now show that certain mixtures of exponentials are HID also.

Theorem 2.2. *If the LT $\phi(\lambda)$ is a mixture of exponentials then $\phi^{\frac{1}{k}}(\lambda)$ is HID and is a mixture of exponentials also.*

Proof. Clearly $\phi(\lambda)$ is GID and hence $\phi^{\frac{1}{k}}(\lambda)$ is HID. From Steutel and van Harn (2004, p.337), if $\phi(\lambda)$ is a mixture of exponentials then $\phi^{\alpha}(\lambda),0<\alpha<1$ is also one. Thus by putting $\alpha=\frac{1}{k}$ we get that $\phi^{\frac{1}{k}}(\lambda)$ is a mixture of exponentials also. $\square$

Theorem 2.3. *Every HID distribution is GID.*

Proof. Let $h(\lambda)$ be the LT that is HID. Then $h(\lambda) = \left(\frac{1}{1+\psi(\lambda)}\right)^{\frac{1}{k}}$, where $\psi(\lambda)$ has CMD and $\psi(0) = 0$. Now, $\left(\frac{1}{1+\psi(\lambda)}\right)^{\frac{1}{k}} = \frac{1}{1+\xi(\lambda)}$, where $\xi(\lambda) = (1 + \psi(\lambda))^{\frac{1}{k}} - 1$. By Feller (1971, p.451) if ψ_1 and ψ_2 are positive functions having CMD then $\psi_1(\psi_2)$ also has CMD. Since $\lambda^{\alpha}, 0 < \alpha < 1$ has CMD, $(1+\lambda)^{\alpha} - 1$ and $\xi(\lambda)$ also have CMDs and further $\xi(0) = 0$. Hence $h(\lambda)$ is GID. $\square$

Remark 2.1. *In a way (for $\alpha = 1/k$) theorems 2.1 and 2.3 together is stronger than the known result, mixtures of gamma(α, β), $0 < \alpha \leq 1$, distributions are ID (Steutel and van Harn, 2004, p.344).*

Using examples, next we prove certain inclusion relations among the classes of HID distributions, GID distributions and mixtures of gamma distributions.

Theorem 2.4. *The class of HID distributions is a proper subclass of the class of GID distributions.*

Proof. Consider the LT $\phi(\lambda) = \left(\frac{1}{1+\lambda}\right)^{\frac{3}{4}} = \frac{1}{1+(1+\lambda)^{\frac{3}{4}}-1} = \frac{1}{1+\xi(\lambda)}$. Now, $\xi(\lambda)$ has CMD, $\xi(0) = 0$ and hence $\phi(\lambda)$ is GID. However, since $\frac{3}{4} \neq \frac{1}{k}$ for $k > 1$ integer, $\phi(\lambda)$ is not HID. The assertion now follows by invoking theorem 2.3. $\square$

Theorem 2.5. *Mixtures of gamma ($\frac{1}{k}, \beta$) distributions, $k > 1$ integer, is a proper subclass of the class of HID distributions.*

Proof. While proving their theorem 2.2, Pillai and Sandhya (1990) showed that the LT $\phi(\lambda) = \frac{1}{1+\ln(1+\lambda)}$ is not a mixture of exponentials but is GID. Hence $\phi^{\frac{1}{k}}(\lambda)$ is HID. However, $\phi^{\frac{1}{k}}(\lambda)$ cannot be a mixture of gamma($\frac{1}{k}, \beta$) distributions because if it were, then by corollary 2.12.2 of Steutel (1970, p.50) $[\phi^{\frac{1}{k}}(\lambda)]^k = \phi(\lambda)$ will be a mixture of exponentials, a contradiction. Thus we have a distribution that is HID but not a mixture of gamma($\frac{1}{k}, \beta$) distributions. By theorem 2.1 the proof is complete. $\square$

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Residual Verma Entropy of k-Record Values

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ABSTRACT

In this paper, we consider a generalized residual entropy known as residual Verma entropy(RVE) of k-record values. A representation of RVE of nth k-record value arising from any continuous distribution is expressed in terms of RVE of nth k-record value arising from uniform distribution. We provide bounds for residual Verma entropy of k-record values. Monotone behaviour of RVE of k-record values in terms of number of observations have also been considered.

Key words and Phrases: *Entropy, k-record values, Verma entropy, residual Verma entropy .*

1 Introduction

Let $\{X_i, i \geq 1\}$ be a sequence of independent and identically distributed (iid) random variables having a common cumulative distribution function (cdf) $F(x)$ which is absolutely continuous. An observation X_j is called an upper record if its value exceeds that of all preceeding observations. Thus, X_j is an upper record if $X_j > X_i$ for every $i < j$. In an analogous way one can define lower record values also.

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Interest in records has increased steadily over the years since its formulation by Chandler (1952). Record value data arise in a wide variety of practical situations. Examples include destructive stress testing, sporting and athletic events, meteorological analysis, oil and mining surveys, hydrology, seismology etc. For a more specific example, consider the situation of testing the breaking strength of wooden beams as described in Glick (1978). For a detailed survey on the theory and applications of record values see, Ahsanullah (2004), Arnold et al. (1998), Nevzorov (2001) and the references therein.

Serious difficulties for the statistical inference procedures based on records arise due to the fact that the occurrences of record data are very rare in practical situations and moreover the expected waiting time is infinite for every record after the first (see, Arnold et al., 1998). To get rid of these limitations one may consider the model of k-record statistics introduced by Dziubdziela and Kopocinski (1976). For a positive integer k, the upper k-record times $T_{n(k)}$ and the upper k-record values $U_{n(k)}$ are defined as follows:

$$T_{1(k)} = k, \quad \text{with probability } 1$$

and, for $n > 1$

$$T_{n(k)} = min\{j : j > T_{n-1(k)} \ , \ X_j > X_{T_{n-1(k)}-k+1:T_{n-1(k)}}\}, \tag{1.1}$$

where $X_{i:m}$ is the i-th order statistic in a random sample of size m. The sequence of upper k-record values are then defined by

$$R_{n(k)} = X_{T_{n(k)}-k+1:T_{n(k)}}, \quad \text{for } n \geq 1. \tag{1.2}$$

In an analogous way, we can define the nth lower k-record times and the nth lower k-record values $L_{n(k)}$. For $k = 1$,the usual classical records are recovered. The pdf of nth upper k-record $R_{n(k)}$ is given by (see Arnold et al., 1998),

$$f_{n(k)}(x) = \frac{k^n}{(n-1)!} \left[-log\{1 - F(x)\}\right]^{n-1} \left[1 - F(x)\right]^{k-1} f(x) \ , \ -\infty < x < \infty. \tag{1.3}$$

The pdf of nth lower k-record $\tilde{R}_{n(k)}$ is given by

$$g_{n(k)}(x) = \frac{k^n}{(n-1)!}\{-logF(x)\}^{n-1}[F(x)]^{k-1}f(x) \ , \ -\infty < x < \infty. \tag{1.4}$$

In reliability theory, the nth upper k-record value is just the failure time of a k-out-of-$T_{n(k)}$ system, where $T_{n(k)}$ is as defined in (1.1). Consider a technical system or subsystem with k-out-of-n structures. A k-out-of-n system breaks down at the time of the $(n-k+1)$th component failure. So, in reliability analysis, the life length of a k-out-of-n system is the $(n-k+1)th$ order statistic in a sample of size n. Consequently, the nth upper k-record value can be regarded as the life length of a k-out-of-$T_{n(k)}$ system. Raqab and Rychlik(2002) presented sharp bounds for the expectations of k-record statistics in various scale units for general distributions. Raqab and Rychlik(2004) established the sharp bounds on the expectations of second record values from symmetric populations. Fashandi and Ahmadi(2006) determined the series approximations for the means of k-record values. For some recent applications of k-record values see, Mary and Chacko (2010) and Chacko and Mary (2013).

Ever since Shannon (1948), has proposed a measure of uncertainty in a discrete distribution based on the Boltzmann entropy, there has been a great deal of interest in the measurement of uncertainty associated with a probability distribution. There are lot of works available in the literature regarding Shannon measure of uncertainty and its applications.Let X be a non-negative random variable with an absolutely continuous distribution with probability density function $f(x)$. Then the Shannon entropy of the random variable X with density f is defined by

$$H(X) = -\int_0^\infty f(x)\log f(x)dx. \tag{1.5}$$

Renyi (1961) developed an entropy of order α of the random variable X and is defined by

$$H_\alpha(X) = \frac{1}{1-\alpha}\log\int_0^\infty [f(x)]^\alpha \, dx, \tag{1.6}$$

where $\alpha > 0(\alpha \neq 1)$.

Clearly

$$\lim_{\alpha\to 1} H_\alpha(X) = H(X) = -\int_0^\infty f(x)\log f(x)dx$$

is the Shannon entropy of X.

There have been several attempts made by researchers to generalize Shannon entropy. Let X be a random variable having an absolutely continuous cdf $F(x)$, survival function $\bar{F}(x)$ and pdf $f(x)$, then Verma entropy of the random variable X with parameters α, β is defined by

$$H_{\alpha}^{\beta}(X) = -\frac{1}{\alpha-\beta} log \int_{-\infty}^{\infty} f^{\alpha+\beta-1}(x)dx, \quad \beta \geq 1, \ \alpha \neq \beta, \ \beta - 1 < \alpha < \beta. \quad (1.7)$$

If $\beta = 1$

$$\begin{aligned} H_{\alpha}^{1}(X) &= -\frac{1}{\alpha-1} log \int_{-\infty}^{\infty} f^{\alpha}(x)dx \\ &= H_{\alpha}(X), \end{aligned}$$

is the Renyi entropy (restricted to $0 < \alpha < 1$), and if $\beta = 1$ and $\alpha \to 1$

$$H(X) = -\int_{-\infty}^{\infty} f(x) log f(x)dx,$$

is the Shannon entropy. Verma entropy plays a vital role as a measure of complexity and uncertainity in different areas such as physics, electronics and engineering to describe many chaotic systems. Richa and Taneja(2012) studied on Verma entropy properties for order statistics. Asha and Chacko(2015) studied on Verma entropy properties of record values.

We often find some situations in practice where the measure defined in () is not an appropriate tool to deal with uncertainty. For example, in reliability and life testing studies, sometimes it is required to modify the current age of a system. Here, one may be interested to study the uncertainty of the random variable $X_t = [X - t/X \geq t]$. The random variable X_t is dubbed as the residual lifetime of a system which has survived upto time $t \geq 0$ and is still working. The residual Verma entropy (RVE), also known as generalized residual entropy of the residual lifetime X_t is given by

$$H_{\alpha}^{\beta}(X,t) = \frac{-1}{\alpha-\beta} log \int_{t}^{\infty} \frac{f^{\alpha+\beta-1}(x)}{\bar{F}^{\alpha+\beta-1}(t)} dx, \quad \beta \geq 1, \ \alpha \neq \beta, \ \beta - 1 < \alpha < \beta. \quad (1.8)$$

which is also known as generalized residual entropy(GRE). (1.8) reduces to Renyi's residual entropy when $\beta = 1$ and reduces to residual entropy when $\beta = 1$ and $\alpha \to 1$. Baig and Dar (2008) studied on generalized residual entropy functions and characterized some life time models based on this measure. Kayal (2014) studied on generalized residual entropy based on order statistics. Kayal (2015c) studied on generalized residual entropy based on record values and weighted distributions. Kayal (2015a,b) studied on generalized dynamic survival entropies of order (α, β).

In this paper, we study some properties of residual Verma entropy based on k-record values arising from any continuous distribution. We obtain the expression for RVE of nth k-record value arising from any continuous distribution in terms of RVE of nth k-record value arising from uniform distribution. Also we study on some properties of generalized residual entropy of k-record values.

2 Residual Verma entropy of k-record values

In this section, we obtain the expression for RVE of k-record value arising from any continuous distribution. For that first we obtain the RVE of nth upper k-record value arising from uniform distribution and is given in the following lemma.

Lemma 2.1. *Let $\{X_i, i \geq 1\}$ be a sequence of iid random variables with a common distribution which is uniform over (0,1). Let $R^*_{n(k)}$ be the nth upper record value arising from the sequence $\{X_i\}$. Then the residual Verma entropy of $R^*_{n(k)}$ is given by*

$$\begin{aligned} H_\alpha^\beta(R^*_{n(k)}, t) &= \frac{-1}{\alpha-\beta} log \frac{\Gamma((n-1)(\alpha+\beta-1)+1, (-log(1-t))((k-1)(\alpha+\beta-1)+1))}{[(k-1)(\alpha+\beta-1)+1]^{(n-1)(\alpha+\beta-1)+1}} \\ &\quad - \frac{1}{\alpha-\beta} log \frac{k^{n(\alpha+\beta-1)}}{\Gamma^{\alpha+\beta-1}(n, -klog(1-t))}. \end{aligned} \tag{2.1}$$

Proof. From the definition of residual Verma entropy, we have

$$\begin{aligned} H_\alpha^\beta(R^*_{n(k)}, t) &= \frac{-1}{\alpha-\beta} log \left(\frac{\int_t^\infty f_{n(k)}^{\alpha+\beta-1}(x)dx}{\bar{F}_{n(k)}^{\alpha+\beta-1}(t)} \right) \\ &= \frac{-1}{\alpha-\beta} log \int_t^1 \left(\frac{k^n}{\Gamma(n)} \right)^{\alpha+\beta-1} [-log(1-x)]^{(n-1)(\alpha+\beta-1)} (1-x)^{(k-1)(\alpha+\beta-1)} dx \\ &\quad + \frac{1}{\alpha-\beta} log \left[\frac{\Gamma(n, -klog(1-t))}{\Gamma(n)} \right]^{\alpha+\beta-1} \\ &= A + B. \end{aligned}$$

Now

$$A = \frac{-1}{\alpha - \beta} log \int_t^1 \left(\frac{k^n}{\Gamma(n)}\right)^{\alpha+\beta-1} [-log(1-x)]^{(n-1)(\alpha+\beta-1)}(1-x)^{(k-1)(\alpha+\beta-1)}dx$$

Putting $z = -log(1-x)$ we get,

$$\begin{aligned} A &= \frac{-1}{\alpha - \beta} log \frac{k^{n(\alpha+\beta-1)}}{\Gamma^{\alpha+\beta-1}(n)} \int_{-log(1-t)}^{\infty} z^{(n-1)(\alpha+\beta-1)} e^{-z[(k-1)(\alpha+\beta-1)+1]}dz \\ &= \frac{-1}{\alpha - \beta} log \left(\frac{k^n}{\Gamma(n)}\right)^{\alpha+\beta-1} \\ &\quad - \frac{1}{\alpha - \beta} log \frac{\Gamma((n-1)(\alpha+\beta-1)+1, -log(1-t)[(k-1)(\alpha+\beta-1)+1])}{[(k-1)(\alpha+\beta-1)+1]^{(n-1)(\alpha+\beta-1)+1}}. \end{aligned}$$

Also

$$B = \frac{1}{\alpha - \beta} log \left[\frac{\Gamma(n, -klog(1-t))}{\Gamma(n)}\right]^{\alpha+\beta-1}.$$

Therefore

$$\begin{aligned} H_\alpha^\beta(R^*_{n(k)}, t) &= A + B \\ &= \frac{-1}{\alpha - \beta} log \left(\frac{k^n}{\Gamma(n)}\right)^{\alpha+\beta-1} \\ &\quad - \frac{1}{\alpha - \beta} log \frac{\Gamma((n-1)(\alpha+\beta-1)+1, -log(1-t)[(k-1)(\alpha+\beta-1)+1])}{[(k-1)(\alpha+\beta-1)+1]^{(n-1)(\alpha+\beta-1)+1}} \\ &\quad + \frac{1}{\alpha - \beta} log \left[\frac{\Gamma(n, -klog(1-t))}{\Gamma(n)}\right]^{\alpha+\beta-1} \\ &= \frac{-1}{\alpha - \beta} log \frac{\Gamma((n-1)(\alpha+\beta-1)+1, (-log(1-t))((k-1)(\alpha+\beta-1)+1))}{[(k-1)(\alpha+\beta-1)+1]^{(n-1)(\alpha+\beta-1)+1}} \\ &\quad - \frac{1}{\alpha - \beta} log \frac{k^{n(\alpha+\beta-1)}}{\Gamma^{\alpha+\beta-1}(n, -klog(1-t))}. \end{aligned}$$

Hence the result.

Theorem 2.1. *Let $\{X_i, i \geq 1\}$ be a sequence of iid continuous random variables with distribution function $F(x)$ and density function $f(x)$. Let $R_{n(k)}$ be the nth upper k-RV of the sequence $\{X_i\}$. Then the RVE of $R_{n(k)}$ is given by*

$$H_\alpha^\beta(R_{n(k)}, t) = H_\alpha^\beta(R^*_{n(k)}, F(t)) - \frac{1}{\alpha - \beta} logE\left[f^{\alpha+\beta-2}F^{-1}(1 - e^{-V_n})\right] \tag{2.2}$$

where $R^*_{n(k)}$ is the nth upper k-RV of uniform distribution over $(0,1)$ and $V_n \sim \Gamma_{-log\bar{F}(t)}((n-1)(\alpha+\beta-1)+1, (k-1)(\alpha+\beta-1)+1)$.

Proof. From the definition of residual Verma entropy, we have

$$
\begin{aligned}
H_{\alpha}^{\beta}(R_{n(k)},t) &= \frac{-1}{\alpha-\beta}log\left(\frac{\int_t^\infty f_{n(k)}^{\alpha+\beta-1}(x)dx}{\bar{F}_{n(k)}^{\alpha+\beta-1}(t)}\right)\\
&= \frac{-1}{\alpha-\beta}log\int_t^\infty\left(\frac{k^n}{\Gamma(n)}\right)^{\alpha+\beta-1}[-log(1-F(x))]^{(n-1)(\alpha+\beta-1)}\\
&\quad (1-F(x))^{(k-1)(\alpha+\beta-1)}(f(x))^{\alpha+\beta-1}dx\\
&\quad +\frac{1}{\alpha-\beta}log\left[\frac{\Gamma(n,-klog(1-F(t)))}{\Gamma(n)}\right]^{\alpha+\beta-1}\\
&= A+B.
\end{aligned}
$$

$$
\begin{aligned}
A &= \frac{-1}{\alpha-\beta}log\int_t^\infty\left(\frac{k^n}{\Gamma(n)}\right)^{\alpha+\beta-1}[-log(1-F(x))]^{(n-1)(\alpha+\beta-1)}\\
&\quad \times(1-F(x))^{(k-1)(\alpha+\beta-1)}(f(x))^{\alpha+\beta-1}dx
\end{aligned}
$$

On putting $z=-log(1-F(x))$, we get

$$
\begin{aligned}
A &= \frac{-1}{\alpha-\beta}log\frac{k^{n(\alpha+\beta-1)}}{\Gamma^{\alpha+\beta-1}(n)}\int_{-log(1-F(t))}^\infty z^{(n-1)(\alpha+\beta-1)}e^{-z[(k-1)(\alpha+\beta-1)+1]}\\
&\quad \times f^{\alpha+\beta-2}(F^{-1}(1-e^{-z}))dz\\
&= \frac{-1}{\alpha-\beta}log\left[\frac{k^n}{\Gamma(n)}\right]^{\alpha+\beta-1}\\
&\quad -\frac{1}{\alpha-\beta}log\frac{\Gamma((n-1)(\alpha+\beta-1)+1,-log(1-F(t))[(k-1)(\alpha+\beta-1)+1])}{[(k-1)(\alpha+\beta-1)+1]^{(n-1)(\alpha+\beta-1)+1}}\\
&\quad -\frac{1}{\alpha-\beta}logE[f^{\alpha+\beta-2}(F^{-1}(1-e^{-V_n}))].
\end{aligned}
$$

Also

$$
B=\frac{1}{\alpha-\beta}log\left[\frac{\Gamma(n,-klog(1-F(t)))}{\Gamma(n)}\right]^{\alpha+\beta-1}.
$$

Therefore

$$\begin{aligned}
H_\alpha^\beta(R_{n(k)},t) &= A+B\\
&= \frac{-1}{\alpha-\beta}log\left[\frac{k^n}{\Gamma(n)}\right]^{\alpha+\beta-1}\\
&\quad -\frac{1}{\alpha-\beta}\\
&\quad log\frac{\Gamma((n-1)(\alpha+\beta-1)+1,-log(1-F(t))[(k-1)(\alpha+\beta-1)+1])}{[(k-1)(\alpha+\beta-1)+1]^{(n-1)(\alpha+\beta-1)+1}}\\
&\quad -\frac{1}{\alpha-\beta}logE[f^{\alpha+\beta-2}(F^{-1}(1-e^{-V_n}))]\\
&\quad +\frac{1}{\alpha-\beta}log\left[\frac{\Gamma(n,-klog(1-F(t)))}{\Gamma(n)}\right]^{\alpha+\beta-1}\\
&= H_\alpha^\beta(R^*_{n(k)},F(t))-\frac{1}{\alpha-\beta}logE\left[f^{\alpha+\beta-2}F^{-1}(1-e^{-V_n})\right].
\end{aligned}$$

Hence the theorem.

Similarly we obtain the expression for RVE of nth lower k-RV arising from any continuous distribution. For that we obtain the residual Verma entropy of lower k-RV arising from uniform distribution and is given in the following lemma.

Lemma 2.2. *Let $\{X_i, i\geq 1\}$ be a sequence of iid random variables with a common distribution which is uniform over $(0,1)$. Let $\tilde{R}^*_{n(k)}$ be the nth lower k-record value arising from the sequence $\{X_i\}$. Then the residual Verma entropy of $\tilde{R}^*_{n(k)}$ is given by*

$$\begin{aligned}
H_\alpha^\beta(\tilde{R}^*_{n(k)},t) &= \frac{-1}{\alpha-\beta}log\frac{\gamma((n-1)(\alpha+\beta-1)+1,-logt[(k-1)(\alpha+\beta-1)+1])}{[(k-1)(\alpha+\beta-1)+1]^{(n-1)(\alpha+\beta-1)+1}}\\
&\quad -\frac{1}{\alpha-\beta}log\frac{k^{n(\alpha+\beta-1)}}{\gamma^{\alpha+\beta-1}(n,-klogt)}. \qquad (2.3)
\end{aligned}$$

The proof of this lemma is omitted as it is similar to that of lemma 2.1.

Theorem 2.2. *Let $X_i, i\geq 1$ be a sequence of iid continuous random variables with distribution function $F(x)$ and density function $f(x)$. Let $\tilde{R}_{n(k)}$ be the nth lower k-RV of the sequence $\{X_i\}$. Then the residual Verma entropy of $\tilde{R}_{n(k)}$ is given by*

$$H_\alpha^\beta(\tilde{R}_{n(k)},t)=H_\alpha^\beta(\tilde{R}^*_{n(k)},F(t))-\frac{1}{\alpha-\beta}logE[f^{\alpha+\beta-2}F^{-1}(e^{-V_n})], \qquad (2.4)$$

where $\tilde{R}^*_{n(k)}$ is the nth lower k-RV of uniform distribution over $(0,1)$ and $V_n\sim \gamma_{-logF(t)}((n-1)(\alpha+\beta-1)+1,(k-1)(\alpha+\beta-1)+1)$.

Proof. From the definition of residual Verma entropy, we have

$$
\begin{aligned}
H_\alpha^\beta(\tilde{R}_{n(k)},t) &= \frac{-1}{\alpha-\beta} log\left(\frac{\int_t^\infty g_{n(k)}^{\alpha+\beta-1}(x)dx}{\bar{G}_{n(k)}^{\alpha+\beta-1}(t)}\right) \\
&= \frac{-1}{\alpha-\beta} log \int_t^\infty \left(\frac{k^n}{\Gamma(n)}\right)^{\alpha+\beta-1} [-logF(x)]^{(n-1)(\alpha+\beta-1)} F(x)^{(k-1)(\alpha+\beta-1)} \\
&\quad (f(x))^{\alpha+\beta-1}dx + \frac{1}{\alpha-\beta} log\bar{G}_{n(k)}^{\alpha+\beta-1}(t) \\
&= A+B.
\end{aligned}
$$

$$
\begin{aligned}
A &= \frac{-1}{\alpha-\beta} log \int_t^\infty \left(\frac{k^n}{\Gamma(n)}\right)^{\alpha+\beta-1} [-logF(x)]^{(n-1)(\alpha+\beta-1)} (F(x))^{(k-1)(\alpha+\beta-1)} \\
&\quad (f(x))^{\alpha+\beta-1}dx
\end{aligned}
$$

Putting $z=-logF(x)$, we get

$$
\begin{aligned}
A &= \frac{-1}{\alpha-\beta} log \left(\frac{k^n}{\Gamma(n)}\right)^{\alpha+\beta-1} \\
&\quad \times \int_{-logF(t)}^0 -z^{(n-1)(\alpha+\beta-1)} e^{-z[(k-1)(\alpha+\beta-1)+1]} f^{\alpha+\beta-2}(F^{-1}e^{-z})dz \\
&= \frac{-1}{\alpha-\beta} log\left[\frac{k^n}{\Gamma(n)}\right]^{\alpha+\beta-1} \\
&\quad -\frac{1}{\alpha-\beta} log \frac{\gamma((n-1)(\alpha+\beta-1)+1, -logF(t)[(k-1)(\alpha+\beta-1)+1])}{[(k-1)(\alpha+\beta-1)+1]^{(n-1)(\alpha+\beta-1)+1}} \\
&\quad -\frac{1}{\alpha-\beta} logE[f^{\alpha+\beta-2}(F^{-1}(e^{-V_n})].
\end{aligned}
$$

Also

$$
B = \frac{1}{\alpha-\beta} log\left[\frac{\gamma(n,-klogF(t))}{\Gamma(n)}\right]^{\alpha+\beta-1}.
$$

Therefore

$$
\begin{aligned}
H_\alpha^\beta(\tilde{R}^*_{n(k)},t) &= A+B \\
&= \frac{-1}{\alpha-\beta} log\left[\frac{k^n}{\Gamma(n)}\right]^{\alpha+\beta-1} \\
&\quad -\frac{1}{\alpha-\beta} log \frac{\gamma((n-1)(\alpha+\beta-1)+1, -logF(t)[(k-1)(\alpha+\beta-1)+1])}{[(k-1)(\alpha+\beta-1)+1]^{(n-1)(\alpha+\beta-1)+1}} \\
&\quad -\frac{1}{\alpha-\beta} logE[f^{\alpha+\beta-2}(F^{-1}(e^{-V_n})] + \frac{1}{\alpha-\beta} log\left[\frac{\gamma(n,-klogF(t))}{\Gamma(n)}\right]^{\alpha+\beta-1} \\
&= H_\alpha^\beta(\tilde{R}^*_{n(k)}, F(t)) - \frac{1}{\alpha-\beta} logE[f^{\alpha+\beta-2}F^{-1}(e^{-V_n})].
\end{aligned}
$$

Hence the theorem.

Example 2.1. *Let $\{X_i, i \geq 1\}$ be a sequence of iid random variables having a common exponential distribution with density function given by*

$$f(x) = \frac{1}{\theta} e^{\frac{-x}{\theta}}, \quad x > 0, \quad \theta > 0.$$

Here

$$F^{-1}(x) = -\theta log(1-x).$$

Now

$$\begin{aligned} E[f^{\alpha+\beta-2}F^{-1}(1-e^{-V_n})] &= \left(\frac{1}{\theta}\right)^{\alpha+\beta-2} \\ &\times \frac{[(k-1)(\alpha+\beta-1)+1]^{(n-1)(\alpha+\beta-1)+1}}{\Gamma((n-1)(\alpha+\beta-1)+1, \frac{1}{\theta}((k-1)(\alpha+\beta-1)+1))} \\ &\frac{\Gamma((n-1)(\alpha+\beta-1)+1, \frac{(\alpha+\beta-1)kt}{\theta})}{[(\alpha+\beta-1)k]^{(n-1)(\alpha+\beta-1)+1}}, \end{aligned}$$

and

$$\begin{aligned} H_\alpha^\beta(R^*_{n(k)}, F(t)) &= \frac{-1}{\alpha-\beta} log\left(\frac{k^n}{\Gamma(n, \frac{kt}{\theta})}\right)^{\alpha+\beta-1} \\ &- \frac{1}{\alpha-\beta} log \frac{\Gamma((n-1)(\alpha+\beta-1)+1, \frac{t}{\theta}((k-1)(\alpha+\beta-1)+1))}{[(k-1)(\alpha+\beta-1)+1]^{(n-1)(\alpha+\beta-1)+1}}. \end{aligned}$$

Then we have,

$$\begin{aligned} H_\alpha^\beta(R_{n(k)}, t) &= H_\alpha^\beta(R^*_{n(k)}, F(t)) - \frac{1}{\alpha-\beta} log E\left[f^{\alpha+\beta-2}F^{-1}(1-e^{-V_n})\right] \\ &= -\frac{1}{\alpha-\beta} log\left[k^{n(\alpha+\beta-1)} \times \left(\frac{1}{\theta}\right)^{\alpha+\beta-2} \times [k(\alpha+\beta-1)]^{(n-1)(\alpha+\beta-1)+1}\right] \\ &\quad -\frac{1}{\alpha-\beta} log \frac{\Gamma((n-1)(\alpha+\beta-1)+1, \frac{kt(\alpha+\beta-1)}{\theta})}{\Gamma^{\alpha+\beta-1}(n, \frac{kt}{\theta})}. \end{aligned}$$

Example 2.2. *Let $\{X_i, i \geq 1\}$ be a sequence of iid random variables having a common Weibull distribution with density function given by*

$$f(x) = \theta\lambda^\theta (x-\mu)^{\theta-1} e^{-(\lambda(x-\mu))^\theta}, \quad x \geq \mu.$$

Here

$$F^{-1}(x) = \mu + \frac{1}{\lambda}[-log(1-x)]^{\frac{1}{\theta}}.$$

Now

$$\begin{aligned} E[f^{\alpha+\beta-2}F^{-1}(1-e^{-V_n})] &= \frac{(\lambda\theta)^{\alpha+\beta-2}[(k-1)(\alpha+\beta-1)+1]^{(n-1)(\alpha+\beta-1)+1}}{[k(\alpha+\beta-1)]^{n(\alpha+\beta-1)+\frac{1}{\theta}(1-(\alpha+\beta-1))}} \\ & \quad \frac{\Gamma(n(\alpha+\beta-1)+\frac{1}{\theta}(1-(\alpha+\beta-1)),k(\alpha+\beta-1)[\lambda(t-\mu)]^{\theta})}{\Gamma((n-1)(\alpha+\beta-1)+1,[\lambda(t-\mu)]^{\theta}((k-1)(\alpha+\beta-1)+1))}, \end{aligned}$$

and

$$\begin{aligned} H_\alpha^\beta(R^*_{n(k)},F(t)) &= \frac{1}{\alpha-\beta}log\Gamma^{\alpha+\beta-1}(n,k(\lambda(t-\mu))^{\theta}) \\ & -\frac{1}{\alpha-\beta}log\Gamma((n-1)(\alpha+\beta-1)+1,(\lambda(t-\mu))^{\theta}(k-1)(\alpha+\beta-1)+1)) \\ & -\frac{n(\alpha+\beta-1)}{\alpha-\beta}log\ k+\frac{(n-1)(\alpha+\beta-1)+1}{\alpha-\beta}log[(k-1)(\alpha+\beta-1)+1]. \end{aligned}$$

Then we have,

$$\begin{aligned} H_\alpha^\beta(R_{n(k)},t) &= \frac{1}{\alpha-\beta}log\frac{\Gamma^{\alpha+\beta-1}(n,k(\lambda(t-\mu))^{\theta})}{\Gamma(n\alpha+\beta-1+\frac{1}{\beta}(1-\alpha+\beta-1),k\alpha+\beta-1(\lambda(t-\mu))^{\theta})} \\ & +\frac{n\alpha+\beta-1+\frac{1}{\beta}(1-\alpha+\beta-1)}{\alpha-\beta}log\ (k(\alpha+\beta-1))-\frac{n(\alpha+\beta-1)}{\alpha-\beta}log\ k \\ & -\frac{(\alpha+\beta-1)-1}{\alpha-\beta}log(\lambda\beta). \end{aligned}$$

3 Properties of residual Verma entropy of k-Record values

In this section, we derive some properties of residual Verma entropy of nth upper and lower k-record values. The following theorem shows the monotone behaviour of residual Verma entropy of the nth upper k-record values in terms of n. In order to prove this we need the following definitions and lemma.

Definition 3.1

(a) Let X and Y be two random variables such that $P\{X > x\} \leq P\{Y > x\}$ for all $x \in (-\infty, \infty)$. Then we say that X is said to be smaller than Y in the usual stochastic order(denoted by $X \leq_{st} Y$).

Shaked and Shanthikumar(1994) showed that $X \leq_{st} Y$ implies that for any non-decreasing(non-increasing) function δ, $E[\delta(X)] \leq (\geq) E[\delta(Y)]$, provided the expectation exists.

(b) Let X and Y be two random variables with densities f and g respectively. The random variable X is said to be smaller than Y in likelihood ratio order (denoted by $X \leq_{lr} Y$) if $f(x)g(y) \geq f(y)g(x)$ for all $x \leq y$.

Note that $X \leq_{lr} Y$ implies $X \leq_{st} Y$ (Bickel and Lehmann, 1976).

Lemma 3.1. *(Kayal, 2014) Let $u(x)$ and $v_\lambda(x)$,$\lambda > 0$ be non-negative functions. Assume that $0 \leq t < c \leq \infty$ and W_λ has a density $f_{W_\lambda}(w)$, where*

$$f_{W_\lambda}(w) = \frac{u^{p\lambda}(w)v_\lambda(w)}{\int_t^c u^{p\lambda}(x)v_\lambda(x)dx},\ w \in (t, c),\ p \in R. \tag{3.1}$$

For $p \in R$, we define a function $h_\gamma(.)$ as

$$h_\gamma(p) = \frac{1}{\beta - \alpha} ln \frac{\int_t^c u^{p\gamma}(x)v_\gamma(x)dx}{\left(\int_t^c u^p(x)v_1(x)dx\right)^\gamma}. \tag{3.2}$$

Then

(i) for $W_\gamma \leq_{st} (\geq_{st}) W_1$, if $u(x)$ is increasing then $h_\gamma(p)$ is a (decreasing) increasing function of p.

(ii) for $W_\gamma \leq_{st} (\geq_{st}) W_1$, if $u(x)$ is decreasing then $h_\gamma(p)$ is an increasing (decreasing) function of p.

Theorem 3.1. *Let $\{X_i, i \geq 1\}$ be a sequence of iid random variables with a common distribution function $F(x)$ and density function $f(x)$. Let $R_{n(k)}$ be the nth upper k-record. Let $f(x)$ be a non-decreasing function. Then $H_\alpha^\beta(R_{n(k),t})$ is non-decreasing (non-increasing) in n if $\alpha + \beta > (<) 2$.*

Proof. We have

$$H_\alpha^\beta(R_{n(k)}, t) = H_\alpha^\beta(R_{n(k)}^*, F(t)) - \frac{1}{\alpha - \beta} log E\left[f^{\alpha+\beta-2}F^{-1}(1 - e^{-V_n})\right]$$

where

$$H_\alpha^\beta(R^*_{n(k)}, F(t)) = \frac{-1}{\alpha-\beta} log \frac{\Gamma((n-1)(\alpha+\beta-1)+1, -log(1-F(t))((k-1)(\alpha+\beta-1)+1))}{[(k-1)(\alpha+\beta-1)+1]^{(n-1)(\alpha+\beta-1)+1}} - \frac{1}{\alpha-\beta} log \frac{k^{n(\alpha+\beta-1)}}{\Gamma^{\alpha+\beta-1}(n, -klog(1-F(t)))}.$$

Now

$$H_\alpha^\beta(R_{n+1(k)}, t) - H_\alpha^\beta(R_{n(k)}, t) = \Delta_\alpha^\beta(n,t) - \frac{1}{\alpha-\beta} log \frac{E\left[f^{\alpha+\beta-2}F^{-1}(1-e^{-V_{n+1}})\right]}{E\left[f^{\alpha+\beta-2}F^{-1}(1-e^{-V_n})\right]}$$

where

$$\Delta_\alpha^\beta(n,t) = H_\alpha^\beta(R^*_{n+1(k)}, F(t)) - H_\alpha^\beta(R^*_{n(k)}, F(t)). \tag{3.3}$$

and $V_n \sim \Gamma_{-log\bar{F}(t)}((n-1)(\alpha+\beta-1)+1, (k-1)(\alpha+\beta-1)+1)$.

Take $u(x) = x$ and $v_\lambda(x) = x^{-(\alpha+\beta-1)}e^{-x[(k-1)(\alpha+\beta-1)+1]}$ in lemma (3.1). Then W_λ has density $f_\lambda(w)$, where

$$f_\lambda(w) = \frac{[(k-1)(\alpha+\beta-1)+1]^{(n-1)(\alpha+\beta-1)+1}}{\Gamma((n-1)(\alpha+\beta-1)+1, -log(1-F(t))((k-1)(\alpha+\beta-1)+1))} \times e^{-w[(k-1)(\alpha+\beta-1)+1]}w^{(n-1)(\alpha+\beta-1)}, \qquad w \in (-log(1-F(t)), \infty).$$

Here both $u(x)$ and $v_\lambda(x)$ are non-negative and $u(x)$ is increasing. Also for $\alpha+\beta >(<)2$

$$h_\lambda = \frac{-1}{\alpha-\beta} log \frac{\int_{-log\bar{F}(t)}^\infty x^{(n-1)(\alpha+\beta-1)}e^{-x[(k-1)(\alpha+\beta-1)+1]}dx}{\left[\int_{-log\bar{F}(t)}^\infty x^{n-1}e^{-kx}dx\right]^{\alpha+\beta-2}}$$

is an increasing (decreasing) function. One can show that $V_n \leq_{lr} V_{n+1}$ and hence $V_n \leq_{st} V_{n+1}$. This implies that for $\alpha+\beta > 2$ $(\alpha+\beta < 2)$,

$$E[f^{\alpha+\beta-2}(F^{-1}(1-e^{-V_n}))] \leq (\geq) E[f^{\alpha+\beta-2}(F^{-1}(1-e^{-V_{n+1}}))]. \tag{3.4}$$

Hence we have

$$\frac{-1}{\alpha-\beta} log \frac{E[f^{\alpha+\beta-2}(F^{-1}(1-e^{-V_{n+1}}))]}{E[f^{\alpha+\beta-2}(F^{-1}(1-e^{-V_n}))]} \geq \ (\leq) 0. \tag{3.5}$$

Thus $H_\alpha^\beta(R_{n(k),t})$ is non-decreasing (non-increasing) in n if $\alpha+\beta > (<)2$ and this completes the theorem.

Theorem 3.2. *Let $\{X_i, i \geq 1\}$ be a sequence of iid random variables with a common distribution function $F(x)$ and density function $f(x)$. Let $\tilde{R}_{n(k)}$ be the nth lower k-RV. Let $f(x)$ be a non-decreasing function. Then $H_\alpha^\beta(\tilde{R}_{n(k),t})$ is non-decreasing (non-increasing) in n if $\alpha+\beta > (<) 2$.*

Proof. We have

$$H_\alpha^\beta(\tilde{R}_{n(k)}, t) = H_\alpha^\beta(\tilde{R}^*_{n(k)}, F(t)) - \frac{1}{\alpha-\beta} log E\left[f^{\alpha+\beta-2} F^{-1}(e^{-V_n})\right]$$

where

$$\begin{aligned} H_\alpha^\beta(\tilde{R}^*_{n(k)}, F(t)) &= \frac{-1}{\alpha-\beta} log \frac{\gamma((n-1)(\alpha+\beta-1)+1, -logF(t)[(k-1)(\alpha+\beta-1)+1])}{[(k-1)(\alpha+\beta-1)+1]^{(n-1)(\alpha+\beta-1)+1}} \\ &\quad - \frac{1}{\alpha-\beta} log \frac{k^{n(\alpha+\beta-1)}}{\gamma^{\alpha+\beta-1}(n, -klogF(t))}. \end{aligned}$$

Now

$$H_\alpha^\beta(\tilde{R}_{n+1(k)}, t) - H_\alpha^\beta(\tilde{R}_{n(k)}, t) = \Delta_\alpha^\beta(n,t) - \frac{1}{\alpha-\beta} log \frac{E\left[f^{\alpha+\beta-2} F^{-1}(e^{-V_{n+1}})\right]}{E\left[f^{\alpha+\beta-2} F^{-1}(e^{-V_n})\right]}$$

where

$$\Delta_\alpha^\beta(n,t) = H_\alpha^\beta(\tilde{R}^*_{n+1(k)}, F(t)) - H_\alpha^\beta(\tilde{R}^*_{n(k)}, F(t)). \tag{3.6}$$

and $V_n \sim \gamma_{-logF(t)}((n-1)(\alpha+\beta-1)+1, (k-1)(\alpha+\beta-1)+1)$.
Take $u(x) = x$ and $v_\lambda(x) = x^{-(\alpha+\beta-1)} e^{-x[(k-1)(\alpha+\beta-1)+1]}$ in lemma (3.1). Then W_λ has density $f_\lambda(w)$, where

$$\begin{aligned} f_\lambda(w) &= \frac{[(k-1)(\alpha+\beta-1)+1]^{(n-1)(\alpha+\beta-1)+1}}{\gamma((n-1)(\alpha+\beta-1)+1, -logF(t)((k-1)(\alpha+\beta-1)+1))} \\ &\quad \times e^{-w[(k-1)(\alpha+\beta-1)+1]} w^{(n-1)(\alpha+\beta-1)}, \qquad w \in (0, -logF(t)). \end{aligned}$$

Here both $u(x)$ and $v_\lambda(x)$ are non-negative and $u(x)$ is increasing. Also for $\alpha+\beta > (<) 2$

$$h_\lambda = \frac{-1}{\alpha-\beta} log \frac{\int_0^{-logF(t)} x^{(n-1)(\alpha+\beta-1)} e^{-x[(k-1)(\alpha+\beta-1)+1]} dx}{\left[\int_0^{-logF(t)} x^{n-1} e^{-kx} dx\right]^{\alpha+\beta-2}}$$

is an increasing (decreasing) function. One can show that $V_n \geq_{lr} V_{n+1}$ and hence $V_n \geq_{st} V_{n+1}$. This implies that for $\alpha+\beta > 2$ ($\alpha+\beta < 2$),

$$E[f^{\alpha+\beta-2}(F^{-1}(e^{-V_n}))] \leq (\geq) E[f^{\alpha+\beta-2}(F^{-1}(e^{-V_{n+1}}))]. \tag{3.7}$$

Hence we have

$$\frac{-1}{\alpha-\beta}log\frac{E[f^{\alpha+\beta-2}(F^{-1}(e^{-V_{n+1}}))]}{E[f^{\alpha+\beta-2}(F^{-1}(e^{-V_n}))]} \geq \ (\leq)0. \tag{3.8}$$

Thus $H_{\alpha}^{\beta}(\tilde{R}_{n(k),t})$ is non-decreasing (non-increasing) in n if $\alpha+\beta > (<)2$ and this completes the theorem.

4 Bounds for residual Verma entropy of k-record values

In this section, we obtain the bounds for RVE of nth upper k-record value arising from any continuous distribution.

Theorem 4.1. *For any random variable X with residual Verma entropy $H_{\alpha}^{\beta}(X) < \infty$ the residual Verma entropy of upper k-record $R_{n(k)}$ is bounded as follows:*
(i) If $m_{r(k)} = max\{(n-1)(\alpha+\beta-1), -log\bar{F}(t)[(k-1)(\alpha+\beta-1)+1]\}$ is the mode of $\Gamma_{-log\bar{F}(t)}((n-1)(\alpha+\beta-1)+1, (k-1)(\alpha+\beta-1)+1)$ then

$$H_{\alpha}^{\beta}(R_{n(k)},t) \leq B_{n(k)}(t) - \frac{1}{\alpha-\beta}log\int_t^{\infty} f^{\alpha+\beta-2}(x)\lambda_F(x)dx$$

where

$$B_{n(k)}(t) = -\frac{1}{\alpha-\beta}log\frac{[(k-1)(\alpha+\beta-1)+1]\times[(n-1)(\alpha+\beta-1)]^{(n-1)(\alpha+\beta-1)}}{\Gamma((n-1)(\alpha+\beta-1), -log\bar{F}(t)[(k-1)(\alpha+\beta-1)+1])} \times e^{-[(n-1)(\alpha+\beta-1)]}$$

and $\lambda_F(x) = \frac{f(x)}{1-F(x)}$ is the hazard rate of X.
(ii) Let $M = f(m) < \infty$ is the mode of X and assume that the assumptions of theorem 2.1 are met. Then for $\alpha+\beta > (<)2$

$$H_{\alpha}^{\beta}(R_{n(k)},t) \leq (\geq) H_{\alpha}^{\beta}(R_{n(k)}^{*}, F(t)) - \frac{\alpha+\beta-2}{\alpha-\beta}logM.$$

Proof. Since $V_n \sim \Gamma_{-log\bar{F}(t)}((n-1)(\alpha+\beta-1)+1, (k-1)(\alpha+\beta-1)+1)$, the mode of this distribution is given by $m = \frac{(n-1)(\alpha+\beta-1)}{(k-1)(\alpha+\beta-1)+1}$.

Also we have,

$$\begin{aligned}
g_{n(k)}(u) &\leq g_{n(k)}(m) \\
&= -\frac{1}{\alpha-\beta} log \frac{[(k-1)(\alpha+\beta-1)+1]\times[(n-1)(\alpha+\beta-1)]^{(n-1)(\alpha+\beta-1)}}{\Gamma((n-1)(\alpha+\beta-1), -log\bar{F}(t)[(k-1)(\alpha+\beta-1)+1])} \\
&\quad \times e^{-[(n-1)(\alpha+\beta-1)]} \\
&= B_{n(k)}(t).
\end{aligned}$$

Now for $\beta-1<\alpha<\beta$,

$$\begin{aligned}
\frac{-1}{\alpha-\beta} logE[f^{\alpha+\beta-2}F^{-1}(1-e^{-V_n})] &= \frac{-1}{\alpha-\beta} log \int_{-log\bar{F}(t)}^{\infty} f^{\alpha+\beta-2}F^{-1}(1-e^{-u}) g_{n(k)}(u)du \\
&\leq \frac{-1}{\alpha-\beta} log \int_{-log\bar{F}(t)}^{\infty} f^{\alpha+\beta-2}F^{-1}(1-e^{-u}) g_{n(k)}(m)du \\
&\leq \frac{-1}{\alpha-\beta} log\ g_{n(k)}(m) \\
&\quad \frac{-1}{\alpha-\beta} log \int_{-log\bar{F}(t)}^{\infty} f^{\alpha+\beta-2}F^{-1}(1-e^{-u})du \\
&\leq B_{n(k)}(t) - \frac{1}{\alpha-\beta} log \int_{-log\bar{F}(t)}^{\infty} f^{\alpha+\beta-2}F^{-1}(1-e^{-u})du.
\end{aligned}$$

On putting $x=F^{-1}(1-e^{-u})$, we get

$$\leq B_{n(k)}(t) - \frac{1}{\alpha-\beta} log \int_{t}^{\infty} f^{\alpha+\beta-2} f^{\alpha+\beta-2}(x)\lambda_F(x)dx.$$

Thus result (i) follows.

Now for $\alpha+\beta>2$,

$$\begin{aligned}
f^{\alpha+\beta-2}F^{-1}(1-e^{-V_n}) &\leq M^{\alpha+\beta-2} \\
E[f^{\alpha+\beta-2}F^{-1}(1-e^{-V_n})] &\leq M^{\alpha+\beta-2} \\
logE[f^{\alpha+\beta-2}F^{-1}(1-e^{-V_n})] &\leq log\ M^{\alpha+\beta-2} \\
\frac{-1}{\alpha-\beta} logE[f^{\alpha+\beta-2}F^{-1}(1-e^{-V_n})] &\leq -\frac{\alpha+\beta-2}{\alpha-\beta} log\ M^{\alpha+\beta-2}.
\end{aligned}$$

Therefore,

$$H_{\alpha}^{\beta}(R_{n(k)}, t) \leq H_{\alpha}^{\beta}(R_{n(k)}^{*}, F(t)) - \frac{\alpha+\beta-2}{\alpha-\beta} logM.$$

Also for $\alpha+\beta<2$,

$$H_{\alpha}^{\beta}(R_{n(k)}, t) \geq H_{\alpha}^{\beta}(R_{n(k)}^{*}, F(t)) - \frac{\alpha+\beta-2}{\alpha-\beta} logM.$$

Thus result (ii) follows.

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On a Generalization of Marshall-Olkin Weibull Distribution and its Applications

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ABSTRACT

In this article, a new generalized four parameter distribution called Marshall-Olkin Weibull Truncated Negative Binomial (MOWTNB) distribution is introduced and studied. Various structural properties of the new distribution are obtained. The distribution of order statistics is also derived. The method of maximum likelihood for estimating the model parameters is discussed. Applications to two real data sets are provided to show the flexibility and potentiality of the new distribution. We have compared the performance of MOWTNB with ten other competitive models, for modelling the two data sets and showed that MOWTNB performs better compared to these existing models. An AR(1) minification model with this distribution as marginal is developed.

Key words and Phrases: *Autoregressive Model, Hazard Rate, Marshall-Olkin Method, Maximum Likelihood Estimate, Minification Process, Order Statistics, Renyi Entropy, Shannon Entropy..*

1 Introduction

Weibull distribution is one of the widely known lifetime distribution that has been extensively used, for modelling data in reliability and survival analysis. One of the

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important property that makes the Weibull distribution so applicable, among other distributions, is that its hazard rates can be decreasing, constant or increasing. However, this distribution is not a good model for describing phenomenon with non-monotone failure rates. Thus, several extensions and modified forms of Weibull distribution have been introduced and studied by various researchers.
The cumulative distribution function (cdf) of Weibull distribution is given by

$$G(x;\beta) = 1 - e^{-x^{\beta}}; \quad x > 0,\ \beta > 0. \tag{1.1}$$

The inclusion of additional parameters to a well-defined distribution has been indicated as a good methodology for providing more flexible new classes of distributions. Marshall and Olkin (1997) introduced a method of adding a parameter to a given baseline model having cdf G(x), thus defined an extended distribution with survival function

$$\bar{F}(x;\alpha) = \frac{\alpha\bar{G}(x)}{G(x) + \alpha\bar{G}(x)}; \quad -\infty < x < \infty,\ \alpha > 0, \tag{1.2}$$

where $\bar{G}(x) = 1 - G(x)$.
Various authors have studied Marshall-Olkin extended models using different baseline models, see Thomas and Jose (2003, 2005), Ghitany *et al.* (2005, 2007), Jose *et al.* (2010), and Ristic and Kundu (2015). Jayakumar and Thomas (2008) introduced a generalization of the family of Marshall-Olkin distributions using Lehman Alternative 1 approach. Tahir and Nadarajah (2015) proposed another generalization of the family of Marshall-Olkin distributions using Lehmann Alternative 2 approach. Nadarajah *et al.* (2013) introduced a generalization of Marshall-Olkin method as follows:
Let X_1, X_2,... be a sequence of independent and identically distributed (i.i.d) random variables with survival function $\bar{G}(x)$ and N be a truncated negative binomial random variable, independent of X_i's, with parameters $\gamma\epsilon(0,1)$ and $\theta > 0$, such that

$$P(N = n) = \frac{\gamma^{\theta}}{1-\gamma^{\theta}} \left(\begin{array}{c} \theta + n - 1 \\ \theta - 1 \end{array} \right) (1-\gamma)^{n}; \quad n = 1, 2, 3, ...$$

If $U_N = \min(X_1, X_2, ..., X_N)$, then the survival function of U_N is

$$\begin{aligned} \bar{F}(x;\alpha) &= \frac{\gamma^\theta}{1-\gamma^\theta}\sum_{n=0}^{\infty}\begin{pmatrix}\theta+n-1\\ \theta-1\end{pmatrix}((1-\gamma)\bar{G}(x))^n \\ &= \frac{\gamma^\theta}{1-\gamma^\theta}[(G(x)+\gamma\bar{G}(x))^{-\theta}-1]; \quad x\epsilon\mathbf{R}. \end{aligned} \tag{1.3}$$

If $\gamma > 1$ and N is a truncated negative binomial random variable with parameters $\frac{1}{\gamma}$ and $\theta > 0$, then $V_N = \max(X_1, X_2, ..., X_N)$ has the same survival function given by equation (1.3).

Jayakumar and Sankaran (2016a) defined a Generalized Uniform distribution using the approach of Nadarajah *et al.* (2013). Babu (2016) introduced Weibull Truncated Negative Binomial distribution. Further, Jayakumar and Sankaran (2017) introduced Generalized Exponential Truncated Negative Binomial distribution and studied its properties .

This paper is organized as follows. In Section 2, we introduce a new generalization of Marshall-Olkin Weibull distribution and discuss the shapes of density function, distribution function and hazard rate function. Various sub-models obtained from the new distribution are presented. The structural properties of the new distribution are derived in Section 3. An algorithm for generation of random numbers from this model is also given in this Section. The distribution of order statistics is derived in Section 4. The maximum likelihood estimation of the model parameters are discussed in Section 5. In Section 6, we analyze two real data sets to illustrate the potential of the proposed distribution. We have compared the performance of MOWTNB with ten other competitive models. A first order autoregressive minification process with new distribution as marginal is developed in Section 7.

2 Marshall-Olkin Weibull Truncated Negative Binomial Distribution

In this section, we introduce a new family of distributions, obtained by substituting the survival function of Marshall-Olkin Weibull distribution

$$\bar{G}(x) = \frac{\alpha e^{-x^\beta}}{1-(1-\alpha)e^{-x^\beta}}; \quad x > 0, \alpha, \beta > 0, \tag{2.1}$$

in the family of distribution given by (1.3).
The new survival function thus obtained is

$$\bar{F}(x;\alpha,\beta,\gamma,\theta) = \frac{\gamma^\theta}{1-\gamma^\theta}\left[\left(\frac{1-(1-\alpha)e^{-x^\beta}}{1-(1-\alpha\gamma)e^{-x^\beta}}\right)^\theta - 1\right], \tag{2.2}$$

for $x > 0$ and $\alpha, \beta, \gamma, \theta > 0$. We refer to this new distribution as the Marshall-Olkin Weibull Truncated Negative Binomial distribution, denoted by MOWTNB$(\alpha, \beta, \gamma, \theta)$. The cdf of MOWTNB$(\alpha, \beta, \gamma, \theta)$ is

$$F(x;\alpha,\beta,\gamma,\theta) = \frac{1}{1-\gamma^\theta} - \frac{\gamma^\theta}{1-\gamma^\theta}\left[\left(\frac{1-(1-\alpha)e^{-x^\beta}}{1-(1-\alpha\gamma)e^{-x^\beta}}\right)^\theta\right]. \tag{2.3}$$

The probability density function of MOWTNB$(\alpha, \beta, \gamma, \theta)$ is

$$f(x;\alpha,\beta,\gamma,\theta) = \frac{\alpha\beta(1-\gamma)\theta\gamma^\theta x^{\beta-1}e^{-x^\beta}}{1-\gamma^\theta}\frac{\left(1-(1-\alpha)e^{-x^\beta}\right)^{\theta-1}}{\left(1-(1-\alpha\gamma)e^{-x^\beta}\right)^{\theta+1}} \tag{2.4}$$

for $x > 0$ and $\alpha, \beta, \gamma, \theta > 0$.

2.1 Special Cases of the MOWTNB distribution

The MOWTNB distribution is a very flexible model that reduces to various models when its parameters assume certain values as listed below.

Case 1. When $\beta = 1$, the MOWTNB distribution reduces to Marshall-Olkin Exponential Truncated Negative Binomial distribution.

Case 2. When $\theta = 1, \gamma \longrightarrow 1$, the MOWTNB distribution reduces to Marshall-Olkin Weibull distribution.

Case 3. When $\beta = \theta = 1, \gamma \longrightarrow 1$, the MOWTNB distribution reduces to Marshall-Olkin Exponential distribution.

Case 4. When $\beta = 2, \theta = 1, \gamma \longrightarrow 1$, the MOWTNB distribution reduces to Marshall-Olkin Rayleigh distribution.

Case 5. When $\beta = \theta = 1, \gamma = 2$, the MOWTNB distribution reduces to Marshall-Olkin Half Logistic distribution.

Case 6. When $\alpha = 1$, the MOWTNB distribution reduces to Weibull Truncated Negative Binomial distribution.

Case 7. When $\alpha = \beta = 1$, the MOWTNB distribution reduces to Exponential Truncated Negative Binomial distribution.

Case 8. When $\alpha = \theta = 1, \gamma \longrightarrow 1$, the MOWTNB distribution reduces to Weibull distribution.

Case 9. When $\alpha = \beta = \theta = 1, \gamma \longrightarrow 1$, the MOWTNB distribution reduces to Exponential distribution.

Case 10. When $\alpha = \theta = 1, \beta = 2, \gamma \longrightarrow 1$, the MOWTNB distribution reduces to Rayleigh distribution.

Case 11. When $\alpha = \beta = \theta = 1, \gamma = 2$, the MOWTNB distribution reduces to Half Logistic distribution.

2.2 Shapes of pdf

Graphs of probability density function for various values of parameters is presented in Figure 1.

2.3 Hazard Function and Reverse Hazard Function

The hazard rate function of MOWTNB$(\alpha, \beta, \gamma, \theta)$ is

$$h(x; \alpha, \beta, \gamma, \theta) = \frac{\alpha\beta(1-\gamma)\theta x^{\beta-1} e^{-x^{\beta}} \left(1-(1-\alpha)e^{-x^{\beta}}\right)^{\theta-1}}{\left[\left(1-(1-\alpha)e^{-x^{\beta}}\right)^{\theta}\left(1-(1-\alpha\gamma)e^{-x^{\beta}}\right) - \left(1-(1-\alpha\gamma)e^{-x^{\beta}}\right)^{\theta+1}\right]}$$

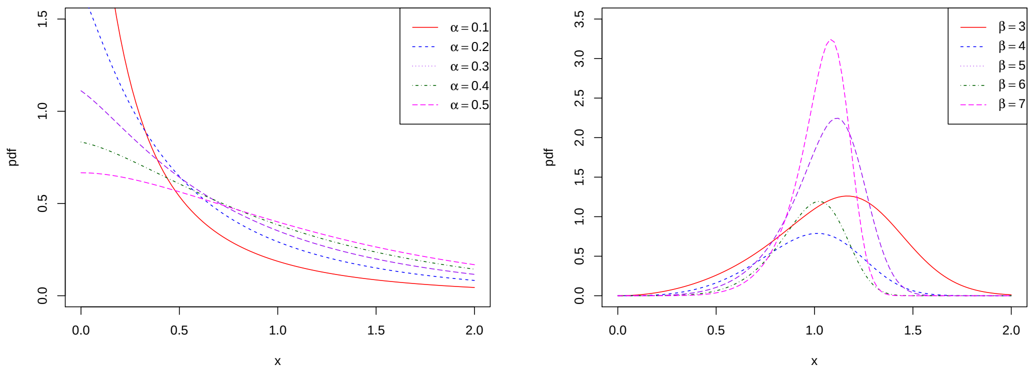

Figure 1: Pdf of MOWTNB$(\alpha,\beta,\gamma,\theta)$ when (i) $\beta = 1, \gamma = 2, \theta = 2$ (left) and (ii) $\alpha = 1.5, \gamma = 2, \theta = 1$ (left), for various values of β.

and the reverse hazard rate function is

$$r(x;\alpha,\beta,\gamma,\theta) = \frac{\alpha\beta(1-\gamma)\theta\gamma^{\theta}x^{\beta-1}e^{-x^{\beta}}\left(1-(1-\alpha)e^{-x^{\beta}}\right)^{\theta-1}}{\left[\left(1-(1-\alpha\gamma)e^{-x^{\beta}}\right)^{\theta+1} - \gamma^{\theta}\left(1-(1-\alpha)e^{-x^{\beta}}\right)^{\theta}\left(1-(1-\alpha\gamma)e^{-x^{\beta}}\right)\right]}.$$

The graphs of the hazard rate of MOWTNB is presented in Figure 2.

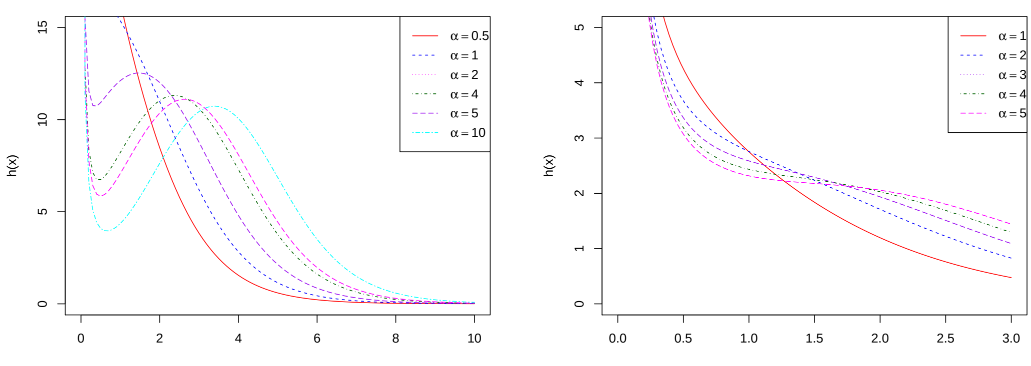

Figure 2: Hazard rate of MOWTNB$(\alpha,\beta,\gamma,\theta)$ when (i) $\beta = 1, \gamma = 10, \theta = 20$ (left) and (ii) $\beta = 1, \gamma = 2, \theta = 10$ (right), for various values of α.

3 Structural Properties

In this section we discuss the statistical properties of MOWTNB distribution.

3.1 Moments

The r^{th} moment of MOWTNB$(\alpha, \beta, \gamma, \theta)$ distribution is given by

$$\begin{aligned}
\mu_r^{'} &= E[X^r] \\
&= \int_0^\infty \frac{\alpha\beta(1-\gamma)\theta\gamma^\theta x^{\beta-1}e^{-x^\beta}}{1-\gamma^\theta} \frac{\left(1-(1-\alpha)e^{-x^\beta}\right)^{\theta-1}}{\left(1-(1-\alpha\gamma)e^{-x^\beta}\right)^{\theta+1}} dx \\
&= \frac{\alpha(1-\gamma)\theta\gamma^\theta}{1-\gamma^\theta} \int_0^1 (-log\ t)^{\frac{r}{\beta}} \frac{(1-(1-\alpha)t)^{\theta-1}}{(1-(1-\alpha\gamma)t)^{\theta+1}} dt,
\end{aligned}$$

by substituting $t = e^{-x^\beta}$.

Case 1.

If $|1-\alpha| < 1, |1-\alpha\gamma| < 1$, then

$$\begin{aligned}
\mu_r^{'} &= \frac{\alpha(1-\gamma)\gamma^\theta\Gamma(\theta+1)}{1-\gamma^\theta} \sum_{i=0}^\infty \sum_{j=0}^\infty \begin{pmatrix} \theta+j \\ \theta \end{pmatrix} \frac{(-1)^j(1-\alpha)^i(1-\alpha\gamma)^j}{\Gamma(\theta-i)i!} \int_0^1 (-log\ t)^{\frac{r}{\beta}} t^{i+j} dt \\
&= \frac{\alpha(1-\gamma)\gamma^\theta\Gamma(\theta+1)\frac{r}{\beta}(\frac{r}{\beta}-1)...(\frac{r}{\beta}-(r-1))}{1-\gamma^\theta} \sum_{i=0}^\infty \sum_{j=0}^\infty \begin{pmatrix} \theta+j \\ \theta \end{pmatrix} \frac{(-1)^j(1-\alpha)^i(1-\alpha\gamma)^j}{\Gamma(\theta-i)i!(i+j+1)^{\frac{r}{\beta}+1}}.
\end{aligned}$$

Case 2. If $|1-\alpha| < \alpha, |1-\alpha\gamma| < \alpha$, by letting $t = 1-u$,

$$\begin{aligned}
\mu_r^{'} &= \frac{\alpha(1-\gamma)\theta\gamma^\theta}{1-\gamma^\theta} \int_0^1 (-log\ (1-u))^{\frac{r}{\beta}} \frac{[1-(1-\alpha)(1-u)]^{\theta-1}}{[1-(1-\alpha\gamma)(1-u)]^{\theta+1}} du, \\
&= \frac{(1-\gamma)\Gamma(\theta+1)\frac{r}{\beta}(\frac{r}{\beta}-1)...(\frac{r}{\beta}-(r-1))}{(1-\gamma^\theta)\alpha\gamma} \\
&\quad \sum_{i=0}^\infty \sum_{j=0}^\infty \sum_{k=0}^{i+j} \begin{pmatrix} \theta+j \\ \theta \end{pmatrix} \frac{(-1)^{j+k}(1-\alpha)^i(1-\alpha\gamma)^j(i+j-k+1)_k}{k!(k+1)^{\frac{r}{\beta}+1}}.
\end{aligned}$$

Table 1 provides the mean, variance, skewness and kurtosis of the MOWTNB distribution for various values of parameters.

Table 1: Mean, Variance, Skewness and Kurtosis of MOWTNB$(\alpha, \beta, \gamma, \theta)$ distribution for different values of α and θ when $\beta = 1, \gamma = 2$.

α	θ	Mean	Variance	Skewness	Kurtosis
1.0	0.1	1.2061	1.2119	1.7254	7.4523
	0.2	1.2257	1.2303	1.7031	7.3399
	0.3	1.2455	1.2485	1.6812	7.2318
1.5	0.1	1.4476	1.4362	1.4804	6.2976
	0.2	1.4696	1.4553	1.4609	6.2154
	0.3	1.4916	1.4740	1.4419	6.1364
2.0	0.1	1.6347	1.5989	1.3263	5.6720
	0.2	1.6581	1.6179	1.3086	5.6063
	0.3	1.6817	1.6366	1.2913	5.5433
2.5	0.1	1.7882	1.7249	1.2167	5.2750
	0.2	1.8128	1.7438	1.2003	5.2200
	0.3	1.8375	1.7623	1.1842	5.1673
3.0	0.1	1.9189	1.8269	1.1330	4.9992
	0.2	1.9444	1.8455	1.1176	4.9517
	0.3	1.9699	1.8636	1.1024	4.9063

3.2 Random Number Generation

Using the method of inversion, we can generate random numbers from the MOWTNB$(\alpha, \beta, \gamma, \theta)$ distribution as

$$X = \left[log \left(\frac{\gamma(1-\alpha) - (1-\alpha\gamma)(1 - Y(1-\gamma^{\theta}))^{\frac{1}{\theta}}}{\gamma - [1 - Y(1-\gamma^{\theta})]^{\frac{1}{\theta}}} \right) \right]^{\frac{1}{\beta}}.$$

where Y∼U(0,1).

3.3 Median

The median of MOWTNB$(\alpha, \beta, \gamma, \theta)$ is given by

$$M = \left[log \left(\frac{2^{\frac{1}{\theta}}\gamma(1-\alpha) - (1-\alpha\gamma)(1+\gamma^{\theta})^{\frac{1}{\theta}}}{2^{\frac{1}{\theta}}\gamma - (1+\gamma^{\theta})^{\frac{1}{\theta}}} \right) \right]^{\frac{1}{\beta}}.$$

3.4 Quantile Function

The quantile function of $X \sim MOWTNB(\alpha, \beta, \gamma, \theta)$ is obtained by inverting Equation (2.3) as

$$x_q = F^{-1}(q) = \left[log \left(\frac{\gamma(1-\alpha) - (1-\alpha\gamma)(1-q(1-\gamma^{\theta}))^{\frac{1}{\theta}}}{\gamma - (1-q(1-\gamma^{\theta}))^{\frac{1}{\theta}}} \right) \right]^{\frac{1}{\beta}}$$

where $F^{-1}(.)$ is the inverse function.

3.5 Rényi Entropy and Shannon Entropy

The Rényi Entropy of MOWTNB$(\alpha, \beta, \gamma, \theta)$ is

$$\begin{aligned} I_R(\eta) &= \frac{1}{1-\eta} log \int_0^{\infty} \left[\frac{\alpha\beta(1-\gamma)\theta\gamma^{\theta} x^{\beta-1} e^{-x^{\beta}}}{1-\gamma^{\theta}} \frac{\left(1-(1-\alpha)e^{-x^{\beta}}\right)^{\theta-1}}{\left(1-(1-\alpha\gamma)e^{-x^{\beta}}\right)^{\theta+1}} \right]^{\eta} dx \\ &= \frac{1}{1-\eta} log \left[\frac{1}{\beta} \left(\frac{\alpha\beta(1-\gamma)\theta\gamma^{\theta}}{1-\gamma^{\theta}} \right)^{\eta} \right] + \\ & \quad \frac{1}{1-\eta} log \left[\int_0^1 t^{\eta-1}(-logt)^{(\eta-1)(\frac{\beta-1}{\beta})} \left[\frac{\left(1-(1-\alpha)e^{-x^{\beta}}\right)^{\theta-1}}{\left(1-(1-\alpha\gamma)e^{-x^{\beta}}\right)^{\theta+1}} \right]^{\eta} dx \right] \end{aligned}$$

The Shannon entropy of X$\sim$ MOWTNB$(\alpha, \beta, \gamma, \theta)$ is given by

$$\begin{aligned} E[-logf(X)] &= E\left[-log \left(\frac{\alpha\beta(1-\gamma)\theta\gamma^{\theta} X^{\beta-1} e^{-X^{\beta}}}{1-\gamma^{\theta}} \frac{\left(1-(1-\alpha)e^{-X^{\beta}}\right)^{\theta-1}}{\left(1-(1-\alpha\gamma)e^{-X^{\beta}}\right)^{\theta+1}} \right) \right] \\ &= log \left[\frac{1-\gamma^{\theta}}{\alpha\beta(1-\gamma)\theta\gamma^{\theta}} \right] - (\beta-1)E[logX] + E[X^{\beta}] \\ & \quad -(\theta-1)E[log(1-(1-\alpha)e^{-X^{\beta}})] + (\theta+1)E[log(1-(1-\alpha\gamma)e^{-X^{\beta}})] \end{aligned}$$

4 Order Statistics

Let $X_1, X_2, ..., X_n$ be a random sample of size n from MOWTNB$(\alpha, \beta, \gamma, \theta)$ and let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ denote the corresponding order statistics. Then, the pdf of i^{th}

order statistic is

$$
\begin{aligned}
f_{X_{(i)}}(x;\alpha,\beta,\gamma,\theta) &= \frac{n!}{(i-1)!(n-i)!} f(x;\alpha,\beta,\gamma,\theta) F^{i-1}(x;\alpha,\beta,\gamma,\theta) \bar{F}^{n-i}(x;\alpha,\beta,\gamma,\theta) \\
&= \frac{n!}{(i-1)!(n-i)!} \frac{\alpha\beta(1-\gamma)\theta\gamma^{\theta}x^{\beta-1}e^{-x^{\beta}}}{1-\gamma^{\theta}} \frac{\left(1-(1-\alpha)e^{-x^{\beta}}\right)^{\theta-1}}{\left(1-(1-\alpha\gamma)e^{-x^{\beta}}\right)^{\theta+1}} \\
&\quad \left[\frac{1}{1-\gamma^{\theta}} - \frac{\gamma^{\theta}}{1-\gamma^{\theta}}\left[\left(\frac{1-(1-\alpha)e^{-x^{\beta}}}{1-(1-\alpha\gamma)e^{-x^{\beta}}}\right)^{\theta}\right]\right]^{i-1} \\
&\quad \left[\frac{\gamma^{\theta}}{1-\gamma^{\theta}}\left[\left(\frac{1-(1-\alpha)e^{-x^{\beta}}}{1-(1-\alpha\gamma)e^{-x^{\beta}}}\right)^{\theta} - 1\right]\right]^{n-i} \\
&= \frac{n!}{(i-1)!(n-i)!} \frac{\alpha\beta\theta\gamma^{(n+1-i)\theta}(1-\gamma)}{(1-\gamma^{\theta})^{n}} x^{\beta-1}e^{-x^{\beta}}(1-(1-\alpha)e^{-x^{\beta}})^{\theta-1} \\
&\quad \left[(1-(1-\alpha\gamma)e^{-x^{\beta}})^{\theta} - \gamma^{\theta}(1-(1-\alpha)e^{-x^{\beta}})^{\theta}\right]^{i-1} \\
&\quad \frac{\left[(1-(1-\alpha)e^{-x^{\beta}})^{\theta} - (1-(1-\alpha\gamma)e^{-x^{\beta}})^{\theta}\right]^{n-i}}{\left[(1-(1-\alpha\gamma)e^{-x^{\beta}})\right]^{1+n\theta}}.
\end{aligned}
$$

The pdf of the largest order statistic, $X_{(n)}$, is

$$
\begin{aligned}
f_{X_{(n)}}(x;\alpha,\beta,\gamma,\theta) &= \frac{n\alpha\beta(1-\gamma)\theta\gamma^{\theta}x^{\beta-1}e^{x^{-\beta}}\left[1-(1-\alpha)e^{-x^{\beta}})\right]^{\theta-1}}{(1-\gamma^{\theta})^{n}} \\
&\quad \frac{\left[\left((1-(1-\alpha\gamma)e^{-x^{\beta}})\right)^{\theta} - \gamma^{\theta}\left((1-(1-\alpha)e^{-x^{\beta}})\right)^{\theta}\right]^{n-1}}{\left[(1-(1-\alpha\gamma)e^{-x^{\beta}})\right]^{1+n\theta}}.
\end{aligned}
$$

The pdf of the smallest order statistic, $X_{(1)}$, is

$$
\begin{aligned}
f_{X_{(1)}}(x;\alpha,\beta,\gamma,\theta) &= \frac{n\alpha\beta(1-\gamma)\theta\gamma^{\theta}x^{\beta-1}e^{x^{-\beta}}\left[1-(1-\alpha)e^{-x^{\beta}})\right]^{\theta-1}}{(1-\gamma^{\theta})^{n}} \\
&\quad \frac{\left[\left((1-(1-\alpha)e^{-x^{\beta}})\right)^{\theta} - \left((1-(1-\alpha\gamma)e^{-x^{\beta}})\right)^{\theta}\right]^{n-1}}{\left[(1-(1-\alpha\gamma)e^{-x^{\beta}})\right]^{1+n\theta}}.
\end{aligned}
$$

5 Estimation of Parameters

The maximum likelihood estimators (MLEs) for the parameters of the MOWTNB distribution are discussed in this section.

Let $x_1, x_2, ..., x_n$ be an observed random sample from MOWTNB$(\alpha, \beta, \gamma, \theta)$ with unknown parameter vector $\nu = (\alpha, \beta, \gamma, \theta)^T$. Then, the likelihood function is

$$\begin{aligned}\ell(x;\nu) &= \prod_{i=1}^{n} f(x;\alpha,\beta,\gamma,\theta) \\ &= \frac{\alpha^n\beta^n\theta^n\gamma^{n\theta}(1-\gamma)^n}{(1-\gamma^\theta)^n}\frac{(\prod_{i=1}^{n} x_i)^{\beta-1}e^{-\sum x_i^\beta}\left[\prod_{i=1}^{n}(1-(1-\alpha)e^{-x_i^\beta})\right]^{\theta-1}}{\left[\prod_{i=1}^{n}(1-(1-\alpha\gamma)e^{-x_i^\beta})\right]^{\theta+1}},\end{aligned}$$

so that the log-likelihood function becomes

$$\begin{aligned}log\ \ell &= nlog\alpha + nlog\beta + nlog\theta + nlog(1-\gamma) + n\theta log\gamma - nlog(1-\gamma^{\theta)} + (\beta-1)\sum_{i=1}^{n} logx_i \\ &\quad -\sum_{i=1}^{n} x_i^\beta + (\theta-1)\sum_{i=1}^{n} log(1-(1-\alpha)e^{-x_i^\beta}) - (\theta+1)\sum_{i=1}^{n} log(1-(1-\alpha\gamma)e^{-x_i^\beta})\end{aligned}$$

Hence the score vector is

$$U(\nu) = \left(\frac{\partial log\ell}{\partial\alpha}, \frac{\partial log\ell}{\partial\beta}, \frac{\partial log\ell}{\partial\gamma}, \frac{\partial log\ell}{\partial\theta}\right)^T.$$

The partial derivatives of log-likelihood function with respect to the parameters are

$$\begin{aligned}\frac{\partial log\ \ell}{\partial\alpha} &= \frac{n}{\alpha} + (\theta-1)\sum_{i=1}^{n}\frac{e^{-x_i^\beta}}{(1-(1-\alpha)e^{-x_i^\beta})} - (\theta+1)\sum_{i=1}^{n}\frac{\gamma e^{-x_i^\beta}}{(1-(1-\alpha\gamma)e^{-x_i^\beta})}, \\ \frac{\partial log\ \ell}{\partial\beta} &= \frac{n}{\beta} + \sum_{i=1}^{n} logx_i - \sum_{i=1}^{n} x_i^\beta logx_i + (\theta-1)\sum_{i=1}^{n}\frac{(1-\alpha)x_i^\beta e^{-x_i^\beta} logx_i}{(1-(1-\alpha)e^{-x_i^\beta})} \\ &\quad -(\theta+1)\sum_{i=1}^{n}\frac{(1-\alpha\gamma)x_i^\beta e^{-x_i^\beta} logx_i}{(1-(1-\alpha\gamma)e^{-x_i^\beta})}, \\ \frac{\partial log\ \ell}{\partial\gamma} &= \frac{-n}{1-\gamma} + \frac{n\theta}{\gamma} + \frac{n\theta\gamma^{\theta-1}}{1-\gamma^\theta} - (\theta+1)\sum_{i=1}^{n}\frac{\alpha e^{-x_i^\beta}}{1-(1-\alpha\gamma)e^{-x_i^\beta}}, \\ \frac{\partial log\ \ell}{\partial\theta} &= nlog\gamma + \frac{n}{\theta} + \frac{n\gamma^\theta log\gamma}{1-\gamma^\theta} + \sum_{i=1}^{n} log(1-(1-\alpha)e^{-x_i^\beta}) - \sum_{i=1}^{n} log(1-(1-\alpha\gamma)e^{-x_i^\beta}).\end{aligned}$$

We can find the estimates of the unknown parameters by setting the score vector equal to zero, $U(\nu) = 0$ and and solving them simultaneously to obtain the ML estimators $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and $\hat{\theta}$. These equations cannot be solved analytically and statistical software can be used to solve them numerically by means of iterative techniques. For the four-parameter MOWTNB distribution, all the second order derivatives exist.
For interval estimation and hypothesis tests of the model parameters, we require the Fisher's information matrix given by

$$I(\nu) \quad = \quad -E \begin{bmatrix} \frac{\partial^2 log\ell}{\partial\alpha^2} & \frac{\partial^2 log\ell}{\partial\alpha\partial\beta} & \frac{\partial^2 log\ell}{\partial\alpha\partial\gamma} & \frac{\partial^2 log\ell}{\partial\alpha\partial\theta} \\ \\ \frac{\partial^2 log\ell}{\partial\alpha\partial\beta} & \frac{\partial^2 log\ell}{\partial\beta^2} & \frac{\partial^2 log\ell}{\partial\beta\partial\gamma} & \frac{\partial^2 log\ell}{\partial\beta\partial\theta} \\ \\ \frac{\partial^2 log\ell}{\partial\alpha\partial\gamma} & \frac{\partial^2 log\ell}{\partial\beta\partial\gamma} & \frac{\partial^2 log\ell}{\partial\gamma^2} & \frac{\partial^2 log\ell}{\partial\gamma\partial\theta} \\ \\ \frac{\partial^2 log\ell}{\partial\alpha\partial\theta} & \frac{\partial^2 log\ell}{\partial\beta\partial\theta} & \frac{\partial^2 log\ell}{\partial\gamma\partial\theta} & \frac{\partial^2 log\ell}{\partial\theta^2} \end{bmatrix}$$

and hence the variance covariance matrix would be $I^{-1}(\nu)$. Clearly, the MOWTNB family satisfies the regularity conditions of MLEs, which are fulfilled for parameters in the interior parameter space but not on the boundary. Hence, the MLE vector $\hat{\nu}$ is consistent and asymptotically normal. That is, $\sqrt{I(\theta)}(\hat{\nu} - \nu)$ converges in distribution to multivariate normal with zero mean vector and identity covariance matrix.
We can use the normal distribution of $\hat{\nu}$ to construct approximate confidence regions for some parameters. The asymptotic $100(1-\xi)\%$ confidence interval for the parameters α, β, γ and θ can be determined as $\hat{\alpha} \pm Z_{\frac{\xi}{2}}\sqrt{V(\hat{\alpha})}$,$\hat{\beta} \pm Z_{\frac{\xi}{2}}\sqrt{V(\hat{\beta})}$,$\hat{\gamma} \pm Z_{\frac{\xi}{2}}\sqrt{V(\hat{\gamma})}$ and $\hat{\theta} \pm Z_{\frac{\xi}{2}}\sqrt{V(\hat{\theta})}$ respectively, where $V(\hat{\alpha})$, $V(\hat{\beta})$, $V(\hat{\gamma})$ and $V(\hat{\theta})$ are the variances of $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ and $\hat{\theta}$, obtained by the diagonal elements of $I^{-1}(\nu)$, and $Z_{\frac{\xi}{2}}$ is the upper $(\frac{\xi}{2})^{th}$ percentile of standard normal distribution.

6 Data Analysis

In this section, we present applications based on two real data sets to demonstrate the flexibility of MOWTNB distribution. In order to compare the new model with other distributions, we consider the goodness-of-fit criteria such as -log $\hat{\ell}$, Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC) and Kolmogorov-Smirnov (K-S) statistic where $\hat{\ell}$ is the likelihood function evaluated at the maximum likelihood estimates,

$$\begin{aligned} AIC &= 2k - 2log\ \hat{\ell}, \\ AICC &= \frac{2kn}{n-k-1} - 2log\ \hat{\ell}, \\ BIC &= klog\ n - 2log\ \hat{\ell}, \\ HQIC &= 2klog(log\ n) - 2log\ \hat{\ell}, \\ K-S &= Sup_x|F_n(x) - F(x)| \end{aligned}$$

where k is the number of parameters and n is the sample size.

We compare the results of MOWTNB distribution with those of the following distributions:

(a) Weibull (W) with pdf

$$f(x;\beta,\lambda) = \beta\lambda^{\beta}x^{\beta-1}e^{-(\lambda x)^{\beta}}; \quad \beta,\lambda > 0,$$

(b) Exponential (E) with pdf

$$f(x;\alpha) = \alpha e^{-\alpha x}; \quad \alpha > 0,$$

(c) Gamma (G) with pdf

$$f(x;\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma\alpha}x^{\alpha-1}e^{-\beta x}; \quad \alpha,\beta > 0,$$

(d) Generalized Exponential (GE) with pdf

$$f(x;\alpha,\lambda) = \alpha\lambda(1-e^{-\lambda x})^{\alpha-1}; \quad \alpha,\lambda > 0,$$

(e) Marshall-Olkin Weibull (MOW) with pdf

$$f(x;\alpha,\lambda,\beta)=\frac{\alpha\beta\lambda^{\beta}x^{\beta-1}e^{-(\lambda x)^{\beta}}}{(1-(1-\alpha)e^{-(\lambda x)^{\beta}})^{2}};\quad \alpha,\beta,\lambda>0,$$

(f) Marshall-Olkin Exponential (MOE) with pdf

$$f(x;\alpha,\lambda)=\frac{\alpha\lambda e^{-\lambda x}}{(1-(1-\alpha)e^{-\lambda x})^{2}};\quad \alpha,\lambda>0,$$

(g) Exponentiated Weibull (EW) with pdf

$$f(x;\gamma,\lambda,\beta)=\beta\gamma\lambda^{\beta}x^{\beta-1}e^{-(\lambda x)^{\beta}}(1-e^{-(\lambda x)^{\beta}})^{\gamma-1};\quad \gamma,\lambda,\beta>0,$$

(h) Exponentiated Weibull Geometric (EWG) with pdf

$$f(x;\alpha,\beta,\gamma,p)=\frac{\alpha\gamma\beta^{\alpha}(1-p)x^{\alpha-1}e^{-(\beta x)^{\alpha}}(1-e^{-(\beta x)^{\alpha}})^{\gamma-1}}{(1-p(1-e^{-(\beta x)^{\alpha}})^{\gamma})^{2}};\quad \alpha,\beta,\gamma>0,p\epsilon(0,1),$$

(i) Generalized Exponential Geometric (GEG) with pdf

$$f(x;\alpha,\beta,p)=\frac{\alpha\beta(1-p)e^{-\beta x}(1-e^{-\beta x})^{\alpha-1}}{(1-pe^{-\beta x})^{\alpha+1}};\quad \alpha,\beta>0,p\epsilon(0,1),$$

(j) Beta Exponential (BE) with pdf

$$f(x;a,b,\lambda)=\frac{\lambda}{B(a,b)}e^{-b\lambda x}(1-e^{-\lambda x})^{a-1};\quad a,b,\lambda>0,$$

(k) Marshall-Olkin Additive Weibull (MOAW) with pdf

$$f(x;\theta,\alpha,\beta,p)=\frac{p(\alpha\theta x^{\theta-1}+\gamma\beta x^{\beta-1})e^{-(\alpha x^{\theta}+\gamma x^{\beta})}}{[p+(1-p)(1-e^{-(\alpha x^{\theta}+\gamma x^{\beta})})]^{2}};\quad \alpha,\theta,\gamma,\beta,p>0,$$

(l) Weibull Truncated Negative Binomial (WTNB) with pdf

$$f(x;\alpha,\theta,\beta,\lambda)=\frac{(1-\alpha)\theta\alpha^{\theta}\beta\lambda^{\beta}x^{\beta-1}e^{-(\lambda x)^{\beta}}}{(1-\alpha^{\theta})(1-(1-\alpha)e^{-(\lambda x)^{\beta}})^{\theta+1}};\quad \alpha,\theta,\beta,\lambda>0,$$

Also, for comparison purpose, we consider the scale parameter λ so that the corresponding pdf of MOWTNB $(\alpha,\beta,\gamma,\theta,\lambda)$ is given by

$$f(x;\alpha,\beta,\gamma,\theta)=\frac{\alpha\beta(1-\gamma)\theta\gamma^{\theta}\lambda^{\beta}x^{\beta-1}e^{-(\lambda x)^{\beta}}}{1-\gamma^{\theta}}\frac{\left(1-(1-\alpha)e^{-(\lambda x)^{\beta}}\right)^{\theta-1}}{\left(1-(1-\alpha\gamma)e^{-(\lambda x)^{\beta}}\right)^{\theta+1}};\ \alpha,\beta,\gamma,\theta,\lambda>0.$$

The better distribution corresponds to smaller -log $\hat{\ell}$, AIC, AICC, BIC, HQIC and K-S values.

The statistical softwares, R and Wolfram Alpha Mathematica have been used to perform necessary calculations and to produce graphics.

6.1 Data Set 1

The first data set represents failure time of 50 items reported in Aarset(1987). Elbatal and Aryal (2013), Elbatal *et al.* (2016) and Jayakumar and Sankaran (2016b) used Transmuted Additive Weibull distribution, The Additive Weibull Geometric distribution and Discrete Mittag-Leffler Additive Weibull distribution respectively, for analyzing this data. We fitted the MOWTNB distribution to this data and compared the results with various models. The model parameters are estimated by maximizing the log-likelihood function using the *NMaximize* procedure in the symbolic computational package *Mathematica*.
Table 2 provides some descriptive statistics of the data.

Table 2: Descriptive Statistics of Aarset data

n	Min.	Max.	Mean	Median	Var	Q_1	Q_3.
50	0.10	86.00	45.69	48.50	1078.15	13.50	81.25

The MLEs of the model parameters and numerical values of log $\hat{\ell}$, AIC, AICC, BIC, HQIC and K-S statistic are presented in Table 3. Plots of the estimated pdf of the MOWTNB model fitted to this data is given in Figure 3. Since the values of $-log\ \hat{\ell}$, AIC, AICC, BIC, HQIC and K-S are smaller for the MOWTNB distribution compared with those values of the other models, the new distribution seems to be a very competitive model to this data.

6.2 Data Set 2

The second data set represents the average amount of precipitation (rainfall) in inches in 70 United States cities. This data set is available as part of the package **data sets** from the **R** software (2011). Nadarajah *et al.* (2013) analysed this data using Exponential Truncated Negative Binomial distribution.
The descriptive statistics of this data is given in Table 4.

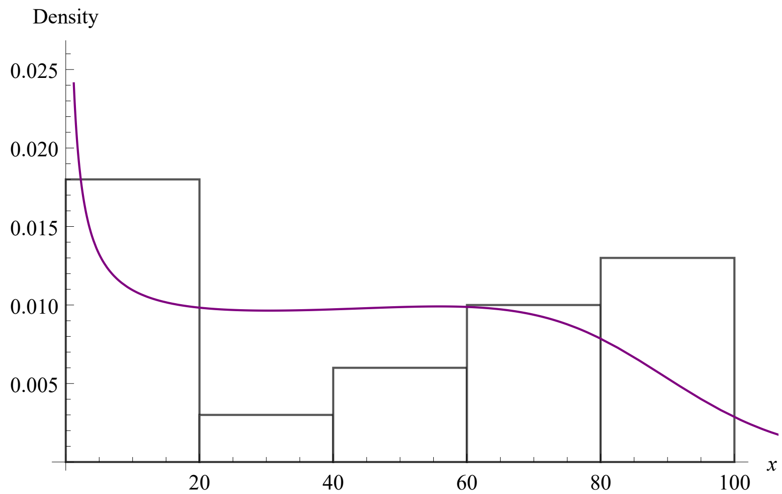

Figure 3: Histogram and fitted pdf for the Aarset data .

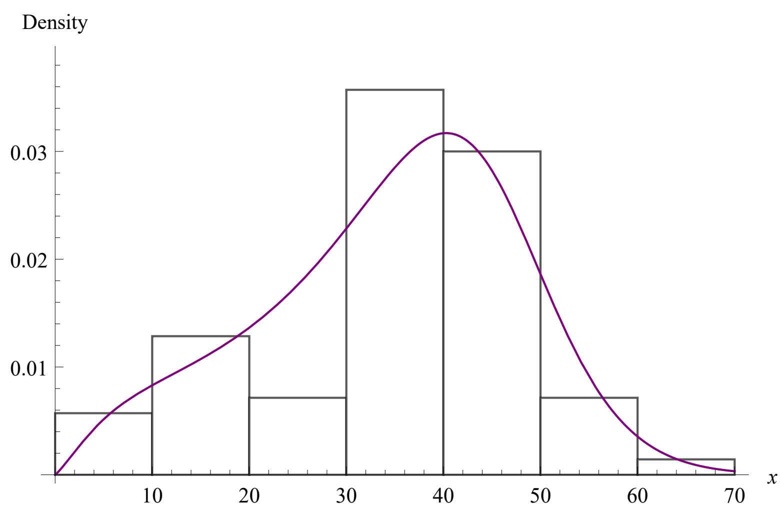

Figure 4: Histogram and fitted pdf for the average amount of precipitation (rainfall) data.

We fitted this data to the new model and compared the results with ten different models. Parameter estimates and numerical values of $-log\ \hat{\ell}$, AIC, AICC, BIC, HQIC and K-S statistic are displayed in Table 5.

Table 3: Parameter estimates and goodness of fit for various models fitted for Aarset data.

Model	Estimates	$-log\ \hat{l}$	AIC	AICC	BIC	HQIC	K-S
MOWTNB	$\hat{\alpha}=0.9024,\ \hat{\beta}=0.5367,$ $\hat{\gamma}=2138.54, \hat{\theta}=0.2016$ $\hat{\lambda}=0.5426$	231.5	472.9	474.3	482.5	476.6	0.07
EWG	$\hat{\gamma}=1.0111,\ \hat{\beta}=0.0259,$ $\hat{\alpha}=1.0689,\ \hat{p}=0.0365$	234.8	477.7	478.6	485.3	480.5	0.14
MOAW	$\hat{\alpha}=0.0002,\ \hat{\gamma}=0.0163,$ $\hat{\theta}=1.9719,\ \hat{\beta}=0.6575,$ $\hat{p}=0.6265$	235.1	480.1	481.5	489.7	483.8	0.16
MOW	$\hat{\lambda}=0.7128,\ \hat{\beta}=0.0682,$ $\hat{\alpha}=6.3727$	238.9	483.8	484.4	489.6	486.0	0.12
MOE	$\hat{\alpha}=2.6215,\ \hat{\lambda}=0.0327,$	239.6	483.1	483.4	486.9	484.6	0.12
GE	$\hat{\alpha}=0.7798,\ \hat{\lambda}=0.0187$	240.0	484.0	484.2	487.8	485.4	0.19
G	$\hat{\alpha}=0.7991,\ \hat{\beta}=0.0175$	240.2	484.4	484.6	488.2	485.8	0.58
W	$\hat{\beta}=0.9480,\ \hat{\lambda}=0.0222$	241.0	486.0	486.2	489.8	490.1	0.17
E	$\hat{\alpha}=0.0218$	241.1	484.1	484.2	486.0	484.9	0.64
WTNB	$\hat{\alpha}=0.5882,\ \hat{\theta}=0.4516,$ $\hat{\beta}=0.9254,\ \hat{\lambda}=0.0261$	245.2	498.4	499.3	506.1	501.3	0.37
EW	$\hat{\gamma}=8.0704,\ \hat{\beta}=782.0999,$ $\hat{\lambda}=0.1626$	261.1	528.1	528.6	533.8	530.4	0.92

Table 4: Descriptive Statistics of the average amount of precipitation (rainfall) data

n	Min.	Max.	Mean	Median	Var	Q_1	Q_3.
70	7.00	67.00	34.89	36.60	187.87	29.38	42.77

Table 5: Parameter estimates and goodness of fit for various models fitted for the average amount of precipitation (rainfall) data.

Model	Estimates	$-log\ \hat{l}$	AIC	AICC	BIC	HQIC	K-S
MOWTNB	$\hat{\alpha} = 0.0589,\ \hat{\beta} = 2.1407,$ $\hat{\gamma} = 379.519, \hat{\theta} = 0.5054$ $\hat{\lambda} = 0.0403$	278.4	566.8	567.8	578.1	571.2	0.09
EW	$\hat{\gamma} = 0.0209,\ \hat{\beta} = 4.9213,$ $\hat{\lambda} = 0.4293$	280.4	566.8	567.2	573.5	569.5	0.11
MOW	$\hat{\alpha} = 2.6809,\ \hat{\beta} = 2.3191,$ $\hat{\lambda} = 0.0319$	280.6	567.1	567.5	573.8	569.9	0.10
MOE	$\hat{\alpha} = 81.0534,\ \hat{\lambda} = 0.1242$	282.0	568.0	568.2	572.5	569.8	0.10
W	$\hat{\beta} = 2.8307,\ \hat{\lambda} = 0.0256$	282.4	568.8	568.9	573.3	571.0	0.13
G	$\hat{\alpha} = 4.7163,\ \hat{\beta} = 0.1352$	288.5	580.9	581.1	585.4	582.8	0.99
BE	$\hat{a} = 4.7103,\ \hat{b} = 13.0026,$ $\hat{\lambda} = 0.0091$	288.6	583.1	583.5	589.8	585.9	0.18
WTNB	$\hat{\alpha} = 0.9566,\ \hat{\theta} = 1.0860,$ $\hat{\beta} = 1.8305,\ \hat{\lambda} = 0.0270$	290.9	589.8	590.4	598.8	593.4	0.17
GE	$\hat{\alpha} = 5.1811,\ \hat{\lambda} = 0.0650$	291.5	586.9	587.1	591.3	588.8	0.19
GEG	$\hat{\alpha} = 5.1814,\ \hat{\beta} = 0.0650$ $\hat{p} = 0.0001$	291.5	588.9	589.3	595.6	591.7	0.19
E	$\hat{\alpha} = 0.0287$	318.6	639.2	639.3	641.5	640.2	1.3

From the plot of estimated pdf presented in Figure 4 and goodness of fit results of the MOWTNB and other models to this data, it clearly indicates that the MOWTNB distribution provides a better fit to this data than the other models.

7 Autoregressive MOWTNB Minification Process

Now, we develop a first order autoregressive (AR(1)) minification process with MOWTNB distribution as marginal distribution.

Consider an AR(1) minification process with structure

$$X_n = \begin{cases} \varepsilon_n & \text{w.p} \quad \rho \\ min(X_{n-1}, \varepsilon_n) & \text{w.p} \quad 1-\rho \end{cases} \qquad 0 < \rho < 1; n \geq 1 \tag{7.1}$$

where $\{\varepsilon_n\}$ is a sequence of i.i.d random variables.

Theorem 7.1. *The AR(1) process given by (7.1), defines a stationary AR(1) minification process with MOWTNB $(\alpha, \beta, \gamma, \theta)$ as marginal distribution if and only if ε_n's are i.i.d MOWTNB $(\rho, \alpha, \beta, \gamma, \theta)$ with $X_0 \stackrel{d}{=} MOWTNB(\alpha, \beta, \gamma, \theta)$.*

Proof. We have, for MOWTNB$(\alpha, \beta, \gamma, \theta)$,

$$\begin{aligned} \bar{F}_X(x) &= \frac{\gamma^\theta}{1-\gamma^\theta}\left[\left(\frac{1-(1-\alpha)e^{-x^\beta}}{1-(1-\alpha\gamma)e^{-x^\beta}}\right)^\theta - 1\right] \\ &= \frac{1}{1+\left[\frac{(1-(1-\alpha\gamma)e^{-x^\beta})^\theta - \gamma^\theta(1-(1-\alpha)e^{-x^\beta})^\theta}{\gamma^\theta[(1-(1-\alpha)e^{-x^\beta})^\theta - (1-(1-\alpha\gamma)e^{-x^\beta})^\theta]}\right]}. \end{aligned}$$

The model (7.1) can be written in terms of survival function as

$$P(X_n > x) = P(\varepsilon_n > x)[\rho + (1-\rho)P(X_{n-1} > x)].$$

That is,

$$\bar{F}_{X_n}(x) = \bar{F}_{\varepsilon_n}(x)[\rho + (1-\rho)\bar{F}_{X_{n-1}}(x)]. \tag{7.2}$$

If $\{X_n\}$ is stationary with MOWTNB $(\alpha, \beta, \gamma, \theta)$ marginals, then

$$\begin{aligned} \bar{F}_{\varepsilon_n}(x) &= \frac{\bar{F}_X(x)}{\rho + (1-\rho)\bar{F}_X(x)} \\ &= \frac{\frac{\gamma^\theta}{1-\gamma^\theta}\left[\left(\frac{1-(1-\alpha)e^{-x^\beta}}{1-(1-\alpha\gamma)e^{-x^\beta}}\right)^\theta - 1\right]}{\rho + (1-\rho)\left\{\frac{\gamma^\theta}{1-\gamma^\theta}\left[\left(\frac{1-(1-\alpha)e^{-x^\beta}}{1-(1-\alpha\gamma)e^{-x^\beta}}\right)^\theta - 1\right]\right\}} \\ &= \frac{1}{1+\rho\left[\frac{(1-(1-\alpha\gamma)e^{-x^\beta})^\theta - \gamma^\theta(1-(1-\alpha)e^{-x^\beta})^\theta}{\gamma^\theta[(1-(1-\alpha)e^{-x^\beta})^\theta - (1-(1-\alpha\gamma)e^{-x^\beta})^\theta]}\right]}. \end{aligned}$$

That is, ε_n's are i.i.d MOWTNB $(\rho, \alpha, \beta, \gamma, \theta)$.

Conversely, if ε_n's are i.i.d MOWTNB $(\rho, \alpha, \beta, \gamma, \theta)$ with $X_0 \stackrel{d}{=}$ MOWTNB $(\alpha, \beta, \gamma, \theta)$, then from (7.2), we have

$$\begin{aligned}
\bar{F}_{X_1}(x) &= \rho\bar{F}_{\varepsilon_1}(x) + (1-\rho)\bar{F}_{\varepsilon_1}(x)\bar{F}_{X_0}(x) \\
&= \rho\left[\frac{1}{1+\rho\left[\frac{(1-(1-\alpha\gamma)e^{-x^\beta})^\theta-\gamma^\theta(1-(1-\alpha)e^{-x^\beta})^\theta}{\gamma^\theta[(1-(1-\alpha)e^{-x^\beta})^\theta-(1-(1-\alpha\gamma)e^{-x^\beta})^\theta]}\right]}\right] + (1-\rho) \\
&\quad \left[\frac{1}{1+\rho\left[\frac{(1-(1-\alpha\gamma)e^{-x^\beta})^\theta-\gamma^\theta(1-(1-\alpha)e^{-x^\beta})^\theta}{\gamma^\theta[(1-(1-\alpha)e^{-x^\beta})^\theta-(1-(1-\alpha\gamma)e^{-x^\beta})^\theta]}\right]}\right]\left[\frac{1}{1+\left[\frac{(1-(1-\alpha\gamma)e^{-x^\beta})^\theta-\gamma^\theta(1-(1-\alpha)e^{-x^\beta})^\theta}{\gamma^\theta[(1-(1-\alpha)e^{-x^\beta})^\theta-(1-(1-\alpha\gamma)e^{-x^\beta})^\theta]}\right]}\right] \\
&= \frac{1}{1+\left[\frac{(1-(1-\alpha\gamma)e^{-x^\beta})^\theta-\gamma^\theta(1-(1-\alpha)e^{-x^\beta})^\theta}{\gamma^\theta[(1-(1-\alpha)e^{-x^\beta})^\theta-(1-(1-\alpha\gamma)e^{-x^\beta})^\theta]}\right]}, \text{on simplification} \\
&= \frac{\gamma^\theta}{1-\gamma^\theta}\left[\left(\frac{1-(1-\alpha)e^{-x^\beta}}{1-(1-\alpha\gamma)e^{-x^\beta}}\right)^\theta - 1\right].
\end{aligned}$$

That is, $X_1 \stackrel{d}{=}$ MOWTNB $(\alpha, \beta, \gamma, \theta)$.

If we assume that $X_{n-1} \stackrel{d}{=}$ MOWTNB$(\alpha, \beta, \gamma, \theta)$, then by induction, we can establish that $X_n \stackrel{d}{=}$ MOWTNB $(\alpha, \beta, \gamma, \theta)$.

Hence the process $\{X_n\}$ is stationary with MOWTNB marginals.

8 Conclusion

In this article, using the approach pioneered by Nadarajah *et al.* (2013), we have introduced a four-parameter model, called the Marshall-Olkin Weibull Truncated Negative Binomial (MOWTNB) distribution, which extends some well-known distributions including Marshall-Olkin Weibull distribution. We have derived various structural properties of MOWTNB distribution. Further, the maximum likelihood estimation of the model parameters are discussed. The new distribution is applied to two real data sets and it provides a better fit than several other competitive models. We hope that the new model will serve as a good alternative to other models available in the literature for modelling positive real data in several areas.

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Generalization of Gompertz Distribution and its Applications in Reliability and Time series

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ABSTRACT

In this paper, as a generalization of the Gompertz distribution, Marshall-Olkin Gompertz distribution is considered. A three parameter $AR(1)$ process is also considered. When X and Y are two independent random variables following Marshall Olkin Gompertz distribution, then average bias, average mean square error, average confidence length and coverage probability of the of the simulated estimates of reliability R is computed. Data analysis based on a real data set and modeling are also done.

Key words and Phrases: *Minification processes; Marshall-Olkin Gompertz distribution; Stress-strength analysis; Reliability* .

1 Introduction

The Gompertz distribution plays an important role in modeling survival times, human mortality and actuarial tables. According to the literature, the Gompertz distribution was formulated by Gompertz (1825) to fit mortality tables. Johnson

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et al. (1995) and Garg et al. (1970) studied the properties of the Gompertz distribution and obtained the maximum likelihood estimates for the parameters. Johnson et al.(1994) note that the Gompertz distribution is a truncated extreme value distribution. Burga et al. (2009) discussed the stress-strength reliability in Gompertz case. The generalized Gompertz distribution is discussed by EL Gohary et al.(2013). Transmuted generalized Gompertz distribution with application is discussed by Muhammad et al. (2017).

New parameters can be introduced to expand families of distributions for added flexibility or to construct covariate models. Introduction of scale parameter usually leads to the accelerated life model, and taking powers of the survival function introduces a parameter that leads to the proportional hazard model. Marshall and Olkin (1997) introduced a method of obtaining an extended family of distributions including one more parameter. For a random variable with a distribution function $F(x)$ and survival function $\overline{F}(x)$, we can obtain a new family of distribution functions called univariate Marshall-Olkin family having cumulative distribution function $G(x)$ given by

$$G(x) = \frac{F(x)}{\alpha + (1-\alpha)F(x)}; -\infty < x < \infty; 0 < \alpha < \infty.$$

Then the corresponding survival function is

$$\overline{G}(x) = \frac{\alpha\overline{F}(x)}{1-(1-\alpha)\overline{F}(x)}; -\infty < x < \infty; 0 < \alpha < \infty.$$

This new family involves an additional parameter α.

In a series of papers Tavares (1980) introduced two stationary markov processes with similar structural form which he had found useful in hydrological applications. In one of these, the observations $\{X_n : n = 0, 1, 2, \ldots\}$ are generated by the equation

$$X_n = K \min(X_{n-1}, \varepsilon_n), n \geq 1, \tag{1.1}$$

where $\{\varepsilon_n, n \geq 1\}$ where $k > 1$ is a constant, and $\{\varepsilon_n\}$ is an innovation process of independently and identically distributed random variables chosen to ensure that $\{X_n\}$ is a stationary markov process with marginal distribution function $F_X(x)$.

Because of the structure of (1) the process $\{X_n\}$ is called a minification process. In this paper, as a generalization of the Gompertz distribution, Marshall-Olkin Gompertz distribution is considered. In section 2, Marshall-Olkin Gompertz distribution is discussed. Some properties of Marshall-Olkin Gompertz distribution are discussed in section 3. Minification processes with Marshall-Olkin Gompertz distribution are developed and discussed in section 4. Estimation of Reliability when X and Y are independent Marshall-Olkin Gompertz distribution is done in section 5. Simulation studies are discussed in section 6. Data analysis and modeling with respect to a real data on TSH of a patient is carried out in section 7. Finally, Conclusions are given in section 8.

2 Marshall-Olkin Gompertz distribution

A random variable X is said to have a Gompertz distribution if its pdf is

$$f_G(x) = \beta e^{\alpha x} e^{-\frac{\beta}{\alpha}(e^{\alpha x}-1)}; x \geq 0, \alpha, \beta > 0$$

Corresponding survival function is

$$\overline{F}_G(x) = e^{-\frac{\beta}{\alpha}(e^{\alpha x}-1)}$$

Hazard rate function is

$$h_G(x) = \beta e^{\alpha x}$$

Apply the Marshall Olkin technique to Gompertz distribution we get the distribution function of the Marshall Olkin Gompertz distribution as

$$G_G(x) = \frac{1 - e^{-\frac{\beta}{\alpha}(e^{\alpha x}-1)}}{1 + (p-1)e^{-\frac{\beta}{\alpha}(e^{\alpha x}-1)}}$$

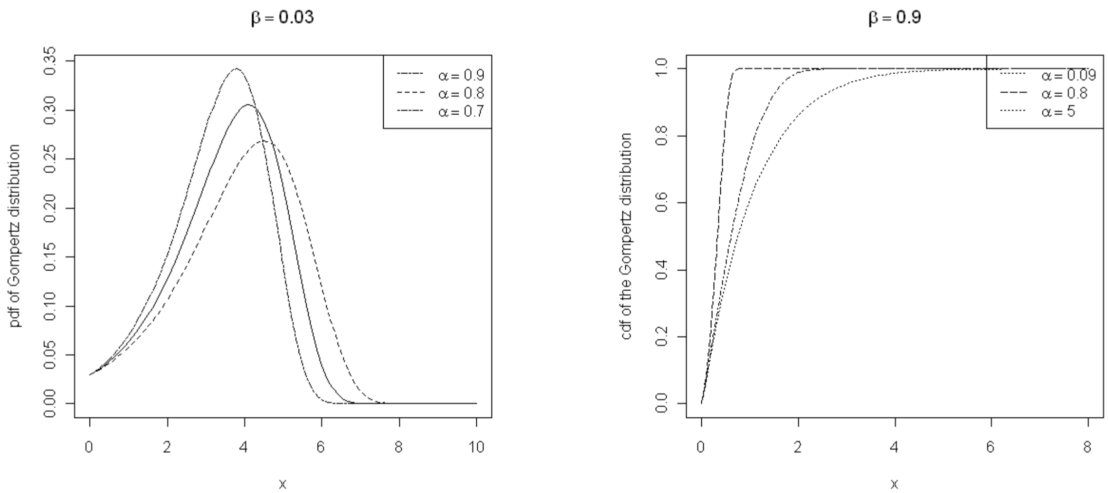
Corresponding pdf is

$$g_G(x) = \frac{p\beta e^{-\frac{\beta}{\alpha}(e^{\alpha x}-1)+\alpha x}}{(1 + (p-1)e^{-\frac{\beta}{\alpha}(e^{\alpha x}-1)})^2}$$

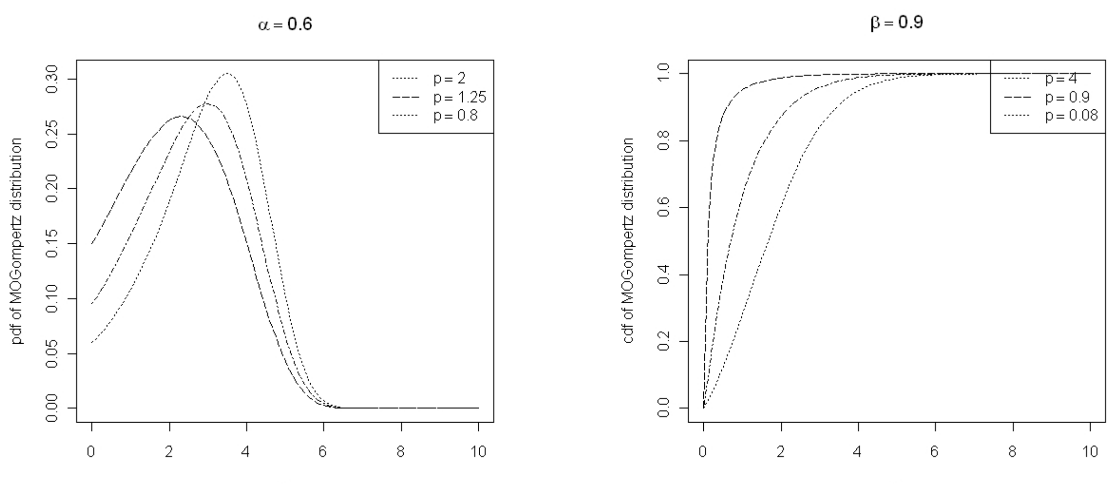
hazard rate function is

$$h_G(x) = \frac{\beta e^{\alpha x}}{1 + (p-1)e^{-\frac{\beta}{\alpha}(e^{\alpha x}-1)}}$$

Graphical Representation of probability density function and distribution function of Gompertz distribution and Marshall-Olkin Gompertz distribution are given below.



Pdf of Gompertz distribution,$\beta = 0.03$,$\alpha = 0.9$,$\alpha = 0.8$ $\alpha = 0.7$; Cdf of Gompertz disti, $\beta = 0.9$,$\alpha = 0.09$,$\alpha = 0.08$,$\alpha = 5$



Pdf of MOGompertz distribution,p=2,p=1.25,p=0.8 $beta = 1.9$,$\alpha = .6$; Cdf of MOGompertz dis,p=4,p=0.4,p=0.08

$\beta = 0.9, \alpha = .9$

Motivation behind this study is due to the following properties attained by the probability distribution by applying Marshall-Olkin technique. It gives added flexibility for the underlying model.It offer a wide range of behavior than the basic distribution from which they are derived. The property that extended distributions can have an interesting hazard function depending on the value of the added parameter. It can be used to model real situation in a better manner than the basic distribution. The geometric minimum stability property of newly formed distri bution can be utilized to develop a stationary process.

3 Characteristic Properties

Definition 3.1. *Let X_1, X_2, … be a sequence of i.i.d. random variables with distribution F in the Marshall-Olkin family and suppose N is independent of the X_i's with a geometric(p) distribution such that*

$$P(N = n) = p(1-p)^{n-1},\ n = 1, 2, \dots$$

Let $U_N = \min(X_1, X_2, \dots, X_N)$ and $V_N = \max(X_1, X_2, \dots, X_N)$. If $F \in \Phi$ implies that the distribution of $U(V)$ is in Φ, then Φ is said to be geometric minimum stable (geometric maximum stable). If Φ is both geometric minimum and geometric maximum stable, then Φ is said to be geometric extreme stable.

Theorem 3.1. *Marshall-Olkin Gompertz distribution is geometric extreme stable.*

Proof. Let X_1, X_2, ... be a sequence of independent identically distributed random variables. Suppose N is independent of the X_i's with a geometric(p) distribution and let $U_N = \min(X_1, X_2, \dots, X_N)$ and $V_N = \max(X_1, X_2, \dots,\ X_N)$. Then we obtain that

$$\bar{G}(x) = P(U_N > x) = \sum_{n=1}^{\infty} \bar{F}^n(x)(1-p)^{n-1}p = \frac{p\bar{F}(x)}{1-(1-p)\bar{F}(x)}\,.$$

Suppose that $\bar{F}$ is the survival function of Marshall-Olkin Gompertz distribution. Then

$$\bar{G}(x) = \frac{pe^{-\frac{\beta}{\alpha}(e^{\alpha x}-1)}}{1+(p-1)e^{-\frac{\beta}{\alpha}(e^{\alpha x}-1)}}$$

and it follows that U_N is geometric minimum stable. Let $V_N = \max(X_1, X_2, \ldots, X_N)$. Then

$$H(x) = P(V_N < x) = \sum_{n=1}^{\infty} F^n(x)(1-p)^{n-1}p = \frac{pF(x)}{1-(1-p)F(x)},$$

so that $\bar{H}(x) = \frac{\bar{F}(x)}{p+(1-p)\bar{F}(x)}$, $x \in R$. If we suppose that $\bar{F}$ is the survival function of Marshall-Olkin Gompertz distribution, then it follows that

$$\bar{H}(x) = \frac{pe^{-\frac{\beta}{\alpha}(e^{\alpha x}-1)}}{h+(p-1)e^{-\frac{\beta}{\alpha}(e^{\alpha x}-1)}}$$

Hence V_N is geometric maximum stable. Thus the family of Marshall-Olkin Gompertz distributions is geometric extreme stable.

4 Autoregressive Minification Processes

Theorem 4.1. *Consider an AR (1) structure given by*

$$X_n = \begin{cases} \epsilon_n & w.p. \quad p(1-q) \\ X_{n-1} & w.p. \quad q \\ \min(pX_{n-1}, \epsilon_n) & w.p. \quad (1-p)(1-q) \end{cases}, \quad n \geq 1. \tag{4.1}$$

where w.p. means with probability. If $q = 0$ we get the ordinary process.

where $0 \leq p \leq 1$, $\{\varepsilon_n\}$ is a sequence of independent and identically distributed random variables independent of $\{X_{n-1}, X_{n-2}, \ldots\}$. Then $\{X_n\}$ is a stationary Markovian AR (1) process with MOG marginals if and only if $\{\varepsilon_n\}$ is distributed as Gompertz distribution.

Proof

From theorem, it follows that

$$\overline{F}_{X_n}(x) = q\overline{F}_{X_{n-1}}(x) + (1-p)(1-q)\overline{F}_{X_{n-1}}(x)\overline{F}_{\varepsilon_n}(x) + p(1-q)\overline{F}_{\varepsilon_n}(x) \tag{4.2}$$

under stationarity

$$\bar{F}_X(x) = \frac{p\bar{F}_\varepsilon(x)}{1-(1-p)\bar{F}_\varepsilon(x)}$$

if we take

$$\bar{F}_\varepsilon(x) = e^{-\frac{\beta}{\alpha}(e^{\alpha x}-1)}$$

Then

$$\bar{F}_X(x) = \frac{pe^{-\frac{\beta}{\alpha}(e^{\alpha x}-1)}}{1-(1-p)(e^{-\frac{\beta}{\alpha}(e^{\alpha x}-1)})}$$

which is the survival function of the Marshall-Olkin Gompertz distribution. Conversely if we take

$$\bar{F}_X(x) = \frac{pe^{-\frac{\beta}{\alpha}(e^{\alpha x}-1)}}{1-(1-p)(e^{-\frac{\beta}{\alpha}(e^{\alpha x}-1)})}$$

then we get

$$\bar{F}_\varepsilon(x) = e^{-\frac{\beta}{\alpha}(e^{\alpha x}-1)}$$

Hence the proof.

5 Estimation of Reliability

Let X and Y be two independent random variables following Marshall Olkin Gompertz distribution with parameters α_1, β, p and α_2, β, p respectively. Then according to Gupta et al (2009) the reliability of the system given by $P(X > Y)$ where X is the strength and Y is the stress is given by

$$\begin{aligned} R = P(X > Y) &= \int_{-\infty}^{\infty} P(X > Y/Y = y) g_Y(y) dy \\ &= \int_0^{\infty} \frac{\alpha_1 \beta e^{-\frac{\beta}{\alpha}(e^{\alpha y}-1)+\alpha y}}{(1+(\alpha_1-1)e^{-\frac{\beta}{\alpha}(e^{\alpha y}-1)})^2} \frac{\alpha_2 e^{-\frac{\beta}{\alpha}(e^{\alpha y}-1)}}{1-(1-\alpha_2)e^{-\frac{\beta}{\alpha}(e^{\alpha y}-1)}} dy \\ &= \frac{\frac{\alpha_1}{\alpha_2}}{(\frac{\alpha_1}{\alpha_2}-1)^2}\left[-\ln\frac{\alpha_1}{\alpha_2} + \frac{\alpha_1}{\alpha_2} - 1\right] \end{aligned}$$

Let $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_n)$ be two independent random samples of sizes m and n from Marshall-Olkin Gompertz distribution with Marshall-Olkin parameters

α_1 and α_2, respectively, and common unknown parameters β and p. L is the log likelihood function, then maximum likelihood estimates of the unknown parameters α_1, α_2 are the solutions of the non-linear equations $\frac{\partial L}{\partial \alpha_1} = 0$ and $\frac{\partial L}{\partial \alpha_2} = 0$ respectively. The elements of information matrix are

$$\begin{aligned} I_{11} &= -E\left(\frac{\partial^2 L}{\partial \alpha_1^2}\right) \\ &= \frac{m}{3\alpha_1^2} \end{aligned}$$

Similarly,

$$I_{22} = -E\left(\frac{\partial^2 L}{\partial \alpha_2^2}\right) = \frac{n}{3\alpha_2^2}$$

$$I_{12} = I_{21} = -E\left(\frac{\partial^2 L}{\partial \alpha_1 \partial \alpha_2}\right) = 0.$$

By the property of m.l.e for $m \to \infty, n \to \infty$, we obtain that

$$(\sqrt{m}(\hat{\alpha_1} - \alpha_1), \sqrt{n}(\hat{\alpha_2} - \alpha_2))^T \xrightarrow{d} N_2\left(\mathbf{0}, \operatorname{diag}\{a_{11}^{-1}, a_{22}^{-1}\}\right),$$

where $a_{11} = \lim\limits_{m,n\to\infty} \dfrac{1}{m} I_{11} = \dfrac{1}{3\alpha_1^2}$ and $a_{22} = \lim\limits_{m,n\to\infty} \dfrac{1}{n} I_{22} = \dfrac{1}{3\alpha_2^2}$. The 95% confidence interval for R is given by

$$\hat{R} \mp 1.96\ \hat{\alpha}_1 b_1(\hat{\alpha}_1, \hat{\alpha}_2)\sqrt{\frac{3}{m} + \frac{3}{n}},$$

where $\hat{R} = R(\hat{\alpha}_1, \hat{\alpha}_2)$ is the estimator of R and

$$b_1(\alpha_1, \alpha_2) = \frac{\partial R}{\partial \alpha_1} = \frac{\alpha_2}{(\alpha_1 - \alpha_2)^3}\left[2(\alpha_1 - \alpha_2) + (\alpha_1 + \alpha_2)\log\frac{\alpha_2}{\alpha_1}\right].$$

6 Simulation Study

We generate $N = 1000$ sets of X-samples and Y-samples from Marshall-Olkin Gompertz distribution with parameters α_1, β, p and α_2, β, p respectively. The combinations of samples of sizes m = 20, 30, 40 and n = 20, 30, 40 are considered. The estimates of α_1 and α_2 are then obtained from each sample to obtain $\hat{R}$. The validity of the estimate of R is discussed by the measures:

Table 1: Average bias and average mean square error of the simulated estimates of R for $\beta = 2, p = 2.5$

	(α_1, α_2)							
	Average bias (b)				**Average Mean Square Error (AMSE)**			
(m,n)	(0.5,2)	(0.7,0.9)	(1.5,1.8)	(1.7,0.7)	(0.5,2)	(0.7,0.9)	(1.5,1.8)	(1.7,0.7)
(20,20)	0.1500	0.0294	0.0210	-0.1019	0.0232	0.0016	0.0010	0.0111
(30,20)	0.1530	0.0302	0.0236	-0.1016	0.0239	0.0016	0.0011	0.0109
(20,30)	0.1515	0.0300	0.0208	-0.1022	0.0236	0.0015	0.0009	0.0110
(20,40)	0.1508	0.0300	0.0217	-0.1035	0.0233	0.0015	0.0009	0.0112
(40,20)	0.1515	0.0301	0.0225	-0.1010	0.0234	0.0015	0.0009	0.0109

1) Average bias of the simulated N estimates of R:

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{R}_i - R)$$

2) Average mean square error of the simulated N estimates of R:

$$\frac{1}{N}\sum_{i=1}^{N}(\hat{R}_i - R)^2$$

3) Average length of the asymptotic 95% confidence intervals of R:

$$\frac{1}{N}\sum_{i=1}^{N} 2(1.96)\hat{\alpha}_{1i}\ b_{1i}(\hat{\alpha}\alpha_{1i}, \hat{\alpha}\alpha_{2i})\sqrt{\frac{3}{m}+\frac{3}{n}}$$

4) The coverage pprobability of the N simulated confidence intervals given by the proportion of such interval that include the parameter R.

7 Data Analysis and Modeling

In this section we analyze a real data set of TSH of a patient (0.02, 0.15, 0.24, 0.30, 0.36, 0.38, 0.40, 0.43, 0.43, 0.46, 0.55, 0.63, 0.84, 0.90, 1.06, 1.13, 1.25, 1.40, 1.44, 1.63, 1.66, 2.16, 2.45, 3.58, 4.14) and compare Marshall Olkin Gompertz distribution

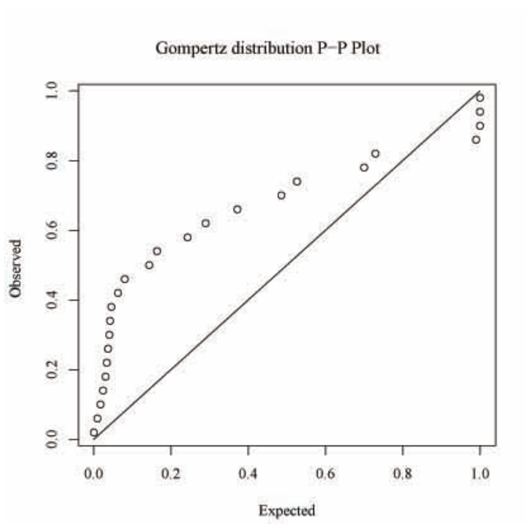
Table 2: Average confidence length and coverage probability of the simulated estimates of R for $\beta = 2, p = 0.5$

	(α_1, α_2)							
	Average confidence length				**Coverage probability**			
(m,n)	(0.5,2)	(0.7,0.9)	(1.5,1.8)	(1.7,0.7)	(0.5,2)	(0.7,0.9)	(1.5,1.8)	(1.7,0.7)
(20,20)	0.3511	0.3567	0.3570	0.3541	0.8530	1	1	0.9960
(30,20)	0.3212	0.3257	0.3260	0.3238	0.6460	1	1	0.9920
(20,30)	0.3209	0.3258	0.3259	0.3246	0.6470	1	1	0.9930
(20,40)	0.3043	0.3091	0.3092	0.3075	0.5250	1	1	0.9910
(40,20)	0.3045	0.3091	0.3093	0.3072	0.5170	1	1	0.9880

with Gompertz distribution.

The P-P plots for the two distributions are given. Estimated values are given in table 3. From that we can conclude that the Marshall Olkin Gompertz distribution is a better fit.

The P-P plots for the Gompertz distribution and Marshall-Olkin Gompertz distribution are given below.



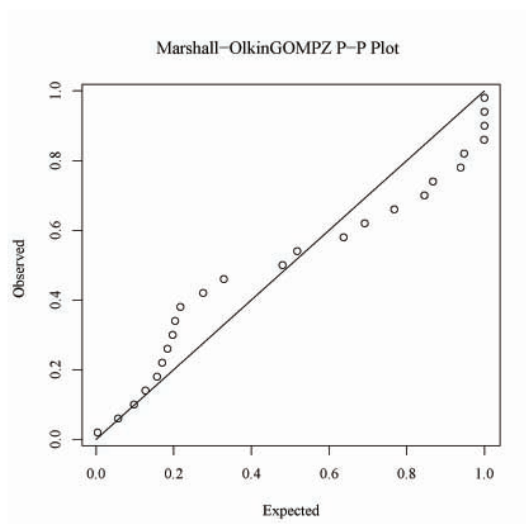


Table:3 Estimated values, loglikelihood, Kolmogrove-Smirnov statistic and P value,AIC,AICc,BIC for the data set are given

Distribution	*Estimates*	$-LogL$	$K-S$	$P-value$	AIC	$AICc$	BIC
Gompertz	$\hat{\alpha}$=2.4634650	75.90857	0.3996	0.0004083	154	154.5	152
	$\hat{\beta}$=0.0550922						
Marshall-Olkin Gompertz	$\hat{\alpha}$=2.0719	69.90571	0.1827	0.3327	144	145.1	142
	$\hat{\beta}$=0.15284 $\hat{p}$=0.45						

8 Conclusion

In this paper Marshall-Olkin Gompertz distribution is considered and its properties are studied. Minification processes with Marshall-Olkin Gompertz marginal distribution are also developed and studied. We analyze a real data set and compare Marshall-Olkin Gompertz distribution with Gompertz distribution. Based on the K-S statistic, P value, AIC, AICc, BIC values and the P-P plot we conclude that Marshall-Olkin Gompertz distribution is a better fit. Estimation of reliability is also done. The new distributions and processes are used for modeling extreme value data on climate changes and environmental statistics.

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