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A General Method of Construction of a Bivariate Lifetime Distribution with a Singular Component

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ABSTRACT

Marshall-Olkin bivariate exponential distribution is the most popular bivariate distribution with a singular component. Since then several other bivariate distributions with a singular component have been introduced in the literature. It is observed that there are mainly two main approaches to construct a bivariate distribution with a singular component. In this paper we have proposed a general method to construct a bivariate distribution with a singular component. All the existing bivariate distributions with a singular component can be obtained using this method. Moreover, more flexible bivariate distributions with a singular component also can be constructed using this method. It is a very simple procedure based on mixing. Using this approach, we have considered one special case, namely bivariate Weibull distribution, in detail. We have derived several properties of the proposed bivariate Weibull distribution and it seems to be more flexible than the popular Marshall-Olkin bivariate Weibull distribution. Maximum likelihood estimators can be obtained quite conveniently in this case. It can be used to model dependent competing risks data and it can be generalized to the multivariate set up also.

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Key words and Phrases: *Marshall-Olkin bivariate exponential distribution; Block and Basu bivariate distributions;maximum likelihood estimators; competing risks.*

1 Introduction

Analyzing bivariate or multivariate data, particularly when they are dependent is a very important problem. It arises in different applications. Several bivariate and multivariate distributions have been proposed in the statistical literature. An extensive amount of work has been done in constructing different bivariate distributions developing their properties and providing various inferential procedures. Some of the well known bivariate distributions are bivariate normal, bivariate log-normal, bivariate-t, bivariate extreme value, bivariate gamma, bivariate exponential, bivariate logistic, bivariate Cauchy, bivariate beta, bivariate skew normal etc. An excellent review of all these bivariate distributions; method of constructions, properties and their applications can be obtained in the book by Balakrishnan and Lai (2009). Some of the recently developed bivariate or multivariate distributions are bivariate Birnbaum-Saunders distribution, bivariate weighted exponential distribution, bivariate generalized exponential distribution etc., see for example Kundu, Balakrishnan and Jamalizadeh (2010), Al-Mutairi, Ghitany and Kundu (2011), Kundu and Gupta (2011) and see the references cited therein.

In all the above cases, the distributions have absolutely continuous cumulative distribution function (CDF). It means that the bivariate distribution function has a proper probability density function (PDF) with respect to a two dimensional Lebesgue measure. Moreover, if X and Y denote the marginals of the bivariate random variable (X, Y), then $P(X = Y) = 0$. Hence if there are ties in the data set, and if it is known that $P(X = Y) > 0$, then none of these distributions can be used to analyze these data sets. In many practical examples it has been observed that there are ties in the data set. It may happen due to truncation, or it may happen due to the physical process by which the data has been obtained, and it is known from the process that $P(X = Y) > 0$. Hence, to analyze these data sets we need a bivariate model so that $P(X = Y) > 0$.

Marshall and Olkin (1967) first introduced such a model, and popularly it is known as the Marshall-Olkin bivariate exponential (MOBE) model. It is a bivariate distribution where the marginals are exponentials and in this case $P(X = Y) > 0$. It has a very interesting physical interpretation and it has an interesting connection with the homogeneous Poisson process also. In the same paper they have introduced bivariate Weibull model also, where the marginals are Weibull and in this case also $P(X = Y) > 0$. From now on we call this as the Marshall-Olkin bivariate Weibull (MOBW) model. For several properties and inferential issues one is referred to Lu (1989, 1992), Kundu and Dey (2009), Kundu and Gupta (2013) and see the references cited therein.

Several other such bivariate distributions have been introduced in the literature. For example, Barreto-Souza and Lemonte (2013) introduced bivariate Kumaraswamy (BVK), bivariate Pareto (BVP), bivariate double generalized exponential (BDGE), bivariate exponentiated Frechet (BEF), bivariate Gumbel (BVG) distributions etc. Kundu and Gupta (2009) proposed the bivariate generalized exponential (BVGE) distribution. Along the same line Sarhan-Balakrishnan bivariate (SBBV) distribution was suggested by Sarhan and Balakrishnan (2007) and modified Sarhan-Balakrishnan bivariate (MSBB) distribution by Kundu and Gupta (2010). Similarly, Sarhan et al. (2011) proposed bivariate generalized linear failure rate distribution and Muhammed (2016) provided the bivariate inverse Weibull distribution. In all these cases the authors provided the method of constructions, derived several properties and developed inference procedures. See for example the recent article by Franco, Vivo and Kundu (2020) in this respect.

It is observed that there are mainly two methods of constructions and they are mainly (a) minimization approach proposed by Marshall and Olkin (1967) and (b) maximization approach proposed by Kundu and Gupta (2009). We will briefly describe them in Section 2. The aim of this paper is two fold. First we introduce a general method of construction of a bivariate distribution with a singular component. The method is very simple, and it is based on the mixture representation. All the existing bivariate distributions with a singular component can be obtained by using this method. Moreover, other more general bivariate distributions with a singular component can be obtained using the proposed method, which cannot be obtained by using the above two methods. The second aim of this paper is to consider one specific case, namely bivariate Weibull distribution with a singular component, which can be obtained by using the proposed method and discuss its properties. We call it as the BWE distribution. The well known MOBE and MOBW can be obtained as special cases of the proposed BWE distribution. It is observed that the maximum likelihood estimators of the unknown parameters can be obtained quite conveniently. Moreover, the proposed BWE model can be used quite conveniently to model dependent competing risks data.

The rest of the paper is organized as follows. In Section 2, we provide a brief background of the two different constructions of a bivariate distribution with a singular component. The general method is proposed in Section 3. The specific case, namely the BWE distribution is discussed in detail in Section 4. The analyses of two data sets have been presented in Section 5, and finally we conclude the paper in Section 6.

2 Background

In this section we provide briefly both the methods for constructing a bivariate distribution with a singular component can be constructed. We will be using the following notation. In this paper it is assumed that all the univariate random variables have non-negative support, although most of the results are valid for random variables with support on the whole real line also. It is further assumed that all the univariate random variables are absolutely continuous and hence they have proper probability density functions. For a random variable X with parameter θ, the probability density function (PDF), the cumulative distribution function (CDF) and survival function (SF) will be denoted by $f_X(x;\theta)$, $F_X(x;\theta)$ and $S_X(x;\theta)$, respectively. Here the parameter θ can be vector valued also. A random variable X is said to follow an exponential distribution with the parameter λ, if the PDF of X is as follows:

$$f_{EX}(x;\lambda) = \begin{cases} \lambda e^{-\lambda x} & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leq 0. \end{cases} \tag{2.1}$$

A random variable X with the PDF (2.1) will be denoted by EX(λ). A random variable X is said to follow a Weibull distribution with the scale parameter λ and the shape parameter α, if the PDF of X is as follows:

$$f_{WE}(x;\alpha,\lambda) = \begin{cases} \alpha\lambda x^{\alpha-1} e^{-\lambda x^{\alpha}} & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leq 0. \end{cases} \tag{2.2}$$

A random variable X with the PDF (2.2) will be denoted by WE(α, λ). In this paper we will consider another lifetime distribution and it is known as the generalized exponential (GE) distribution. A two-parameter generalized exponential distribution is an extension of the one parameter exponential distribution and it has many properties which are very close to a two-parameter gamma distribution. Since it has a very convenient PDF and CDF it can be used as an alternative to the gamma distribution. For a detailed discussion on GE distribution one is referred to the review article by Nadarajah (2011). A random variable X is said to follow a generalized exponential (GE) distribution with the scale parameter λ and the shape parameter α, if the PDF of X is as follows:

$$f_{GE}(x;\alpha,\lambda) = \begin{cases} \alpha\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1} & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x \leq 0. \end{cases} \tag{2.3}$$

A random variable X with the PDF (2.3) will be denoted by GE(α, λ). Now we are going to define the MOBE, MOBW and BVGE distributions.

2.1 Minimization Approach

Suppose U_1, U_2 and U_3 are three independent random variables, then a bivariate random variable (X, Y) can be constructed as follows:

$$X = \min\{U_1, U_3\} \quad \text{and} \quad Y = \min\{U_2, U_3\}.$$

Marshall and Olkin (1967) first proposed this method and it is assumed that U_1 follows ($\sim$) EX(λ_1), $U_2 \sim$ EX(λ_2) and $U_3 \sim$ EX(λ_3). Hence, the joint survival function, $S_{MOBE}(x,y) = P(X > x, Y > y)$, of (X,Y) for $x > 0, y > 0$, becomes

$$S_{MOBE}(x,y) = P(U_1 > x, U_2 > y, U_3 > z) = \begin{cases} e^{-(\lambda_1+\lambda_3)x-\lambda_2 y} & \text{if} \quad 0 < y < x < \infty \\ e^{-\lambda_1 x-(\lambda_2+\lambda_3)y} & \text{if} \quad 0 < x < y < \infty. \end{cases}$$

Here $z = \max\{x,y\}$. From now on, we call this as the Marshall-Olkin bivariate exponential (MOBE) distribution. The MOBE is not an absolutely continuous distribution, and in this case $P(X = Y) = \dfrac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3} > 0$. It does not have a joint PDF with respect to a two-dimensional Lebesgue measure. It has the following joint PDF with respect to two-dimensional Lebesgue measure on $x \neq y$ and with respect to one-dimensional Lebesgue measure on $x = y$, see for example Bemis, Bain and Higgins (1972). The joint PDF of the MOBE is as follows:

$$f_{MOBE}(x,y) = \frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+\lambda_3} f_{MOBE}^{(ac)}(x,y) + \frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3} f_{MOBE}^{(si)}(x,y),$$

where

$$f_{MOBE}^{(ac)}(x,y) = \begin{cases} \frac{\lambda_1+\lambda_2+\lambda_3}{\lambda_1+\lambda_2} f_{EX}(x;\lambda_1+\lambda_3) f_{EX}(y;\lambda_2) & \text{if} \quad x \geq y \\ \frac{\lambda_1+\lambda_2+\lambda_3}{\lambda_1+\lambda_2} f_{EX}(x;\lambda_1) f_{EX}(y;\lambda_2+\lambda_3) & \text{if} \quad y > x, \end{cases}$$

and

$$f_{MOBE}^{(si)}(x,y) = \begin{cases} f_{EX}(x;\lambda_1+\lambda_2+\lambda_3) & \text{if} \quad x = y \\ 0 & \text{if} \quad x \neq y. \end{cases}$$

It simply means

$$\begin{aligned} S_{MOBE}(x,y) \quad = \quad & \frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+\lambda_3} \int_x^\infty \int_y^\infty f_{MOBE}^{(ac)}(u,v)dudv + \\ & \frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3} \int_z^\infty f_{MOBE}^{(si)}(u,u)du. \end{aligned}$$

It can be easily seen that $f_{MOBE}^{ac}(x,y)$ is a proper bivariate PDF. Now MOBW distribution can be obtained by using Weibull distributions instead of exponential distributions. The MOBW distribution can be defined as follows. Suppose $U_1 \sim$ WE(α, λ_1), $U_2 \sim$ WE(α, λ_2), $U_3 \sim$ WE(α, λ_3), and they are independently distributed, then (X,Y), where

$$X = \min\{U_1, U_3\} \quad \text{and} \quad Y = \min\{U_2, U_3\},$$

is said to have a MOBW distribution. The joint PDF of MOBW can be obtained as given below:

$$f_{MOBW}(x,y) = \frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+\lambda_3} f^{(ac)}_{MOBW}(x,y) + \frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3} f^{(si)}_{MOBW}(x,y),$$

where

$$f^{(ac)}_{MOBW}(x,y) = \begin{cases} \frac{\lambda_1+\lambda_2+\lambda_3}{\lambda_1+\lambda_2} f_{WE}(x;\alpha,\lambda_1+\lambda_3) f_{WE}(y;\alpha,\lambda_2) & \text{if} \quad x > y \\ \frac{\lambda_1+\lambda_2+\lambda_3}{\lambda_1+\lambda_2} f_{WE}(x;\alpha,\lambda_1) f_{WE}(y;\alpha,\lambda_2+\lambda_3) & \text{if} \quad y > x, \end{cases}$$

and

$$f^{(si)}_{MOBW}(x,y) = \begin{cases} f_{WE}(x;\alpha,\lambda_1+\lambda_2+\lambda_3) & \text{if} \quad x = y \\ 0 & \text{if} \quad x \neq y. \end{cases}$$

Several other bivariate distributions, for example Sarhan-Balakrishnan bivariate distribution by Sarhan and Balakrishnan (2007) or modified Sarhan-Balakrishnan bivariate distribution by Kundu and Gupta (2010) have been obtained using the same procedure. Now we will provide the maximization method.

2.2 Maximization Approach

Suppose U_1, U_2 and U_3 are three independent random variables, then based on these three random variables, the following bivariate random variable (X,Y) can be constructed, where

$$X = \max\{U_1,U_3\} \qquad \text{and} \qquad Y = \max\{U_2,U_3\}.$$

Kundu and Gupta (2009) first proposed this method to construct BVGE distribution, and it can be obtained as follows. Suppose $U_1 \sim \text{GE}(\alpha_1,\lambda)$, $U_2 \sim \text{GE}(\alpha_2,\lambda)$, $U_3 \sim \text{GE}(\alpha_3,\lambda)$, and they are independently distributed. They (X,Y) as defined above is said to have a BVGE distribution. The joint CDF of (X,Y) for $x > 0$ and $y > 0$, is

$$\begin{aligned} F_{BVGE}(x,y) &= P(U_1 \le x, U_2 \le y, U_3 \le z) \\ &= \begin{cases} (1-e^{-\lambda x})^{\alpha_1}(1-e^{-\lambda y})^{\alpha_2+\alpha_3} & \text{if} \quad 0 < y < x < \infty \\ (1-e^{-\lambda x})^{\alpha_1+\alpha_3}(1-e^{-\lambda y})^{\alpha_2} & \text{if} \quad 0 < x \le y < \infty, \end{cases} \end{aligned}$$

and the joint PDF becomes:

$$f_{BVGE}(x,y) = \frac{\alpha_1+\alpha_2}{\alpha_1+\alpha_2+\alpha_3} f^{(ac)}_{BVGEW}(x,y) + \frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3} f^{(si)}_{BVGE}(x,y),$$

where

$$f^{(ac)}_{BVGE}(x,y) = \begin{cases} \frac{\alpha_1+\alpha_2+\alpha_3}{\alpha_1+\alpha_2} f_{GE}(x;\alpha_1,\lambda) f_{WE}(y;(\alpha_2+\alpha_3),\lambda) & \text{if} \quad y < x \\ \frac{\alpha_1+\alpha_2+\alpha_3}{\alpha_1+\alpha_2} f_{GE}(x;(\alpha_1+\alpha_3),\lambda) f_{WE}(y;\alpha_2,\lambda) & \text{if} \quad x < y, \end{cases}$$

and

$$f^{(si)}_{BVGE}(x,y) = \begin{cases} f_{GE}(x;\alpha_1+\alpha_2+\alpha_3,\lambda) & \text{if} \quad x = y \\ 0 & \text{if} \quad x \neq y. \end{cases}$$

Several other bivariate distributions with a singular component have been developed by using this method, for example the bivariate inverse Weibull distribution by Muhammed (2016), the bivariate generalized linear failure rate distribution proposed by Sarhan et al. (2011) or a very general proportional reversed hazard bivariate model as suggested by Kundu and Gupta (2010) etc. In the next section we propose a very general method of constructing a bivariate distribution with a singular component using that all these distributions can be obtained as special cases.

3 Proposed Method

A bivariate random variable (X,Y) is said to have a bivariate distribution with a singular component, if the joint PDF of (X,Y) can be written as

$$f_{X,Y}(x,y) = p f^{(ac)}_{X,Y}(x,y) + (1-p) f^{(si)}_{X,Y}(x,y). \tag{3.1}$$

Here, $0 < p < 1$, $f^{(ac)}_{X,Y}(x,y)$ is a proper two dimensional PDF, and $f^{(si)}_{X,Y}(x,x)$ is a proper one dimensional PDF, and $f^{(si)}_{X,Y}(x,y) = 0$, if $x \neq y$. Now it is very clear that with proper choice of p, $f^{(ac)}_{X,Y}(x,y)$ and $f^{(si)}_{X,Y}(x,y)$ it is possible to obtain all the existing bivariate distributions with a singular component.

Some of the advantages of the proposed class of distributions can be described as follows. Note that the bivariate Weibull geometric distribution which has been proposed by Kundu and Gupta (2014) cannot be obtained by the above minimization or maximization approach, but with the proper choice of p, $f_{X,Y}^{(ac)}(x,y)$ and $f_{X,Y}^{(si)}(x,y)$, it is possible to obtain based on the proposed method. It may be mentioned that all the existing bivariate distributions with a singular component have positive correlation coefficient, and it is due to construction. It is not possible to obtain negative correlation between the two variables based on the existing methods. But in our proposed method it is not a restriction. It is possible to obtain correlation on the entire range. For example, we can construct $f_{X,Y}^{(ac)}(x,y)$ based on a Gaussian copula with any marginal distribution function, and it is possible to obtain the correlation on the entire range, namely from (-1,1).

The following interpretation can be given for the proposed model. Suppose U, V and W are three random variables, and (U,V) has a joint PDF $f_{U,V}(u,v)$ and W has the PDF $f_W(w)$. We consider the following bivariate random variable (X,Y) as follows:

$$(X,Y) = \begin{cases} (U,V) & \text{with probability } p \\ (W,W) & \text{with probability } 1-p, \end{cases}$$

then for $0 < p < 1$ and if $f_{U,V}(u,v) = f_{X,Y}^{(ac)}(u,v)$, $f_W(w) = f_{X,Y}^{(si)}(w,w)$, then (X,Y) will have the same joint PDF as in (3.1).

In this section we mainly derive some of the general properties of this proposed bivariate distribution and in the subsequent sections we consider one specific case and discuss its properties. The joint CDF and the joint SF of (X,Y) will be denoted by $F_{X,Y}(x,y)$ and $S_{X,Y}(x,y)$, respectively. The joint CDF and the joint SF corresponding to the joint PDF $f_{X,Y}^{(ac)}(x,y)$ will be denoted by $F_{X,Y}^{(ac)}(x,y)$ and $S_{X,Y}^{(ac)}(x,y)$, respectively. Similarly, the CDF and SF of the PDF $f_{X,Y}^{(si)}(x,x)$ will be denoted by $F_{X,Y}^{(si)}(x)$ and $S_{X,Y}^{(si)}(x)$, respectively. Therefore, the joint CDF and the joint SF of (X,Y) can be written as

$$F_{X,Y}(x,y) = pF_{X,Y}^{(ac)}(x,y) + (1-p)F_{X,Y}^{(si)}(\min\{x,y\}) \quad \text{and} \tag{3.2}$$

$$S_{X,Y}(x,y) = pS_{X,Y}^{(ac)}(x,y) + (1-p)S_{X,Y}^{(si)}(\max\{x,y\}), \tag{3.3}$$

respectively. Hence, the CDF of X and Y can be obtained as

$$\begin{aligned} P(X \le x) &= F_X(x) = pF^{(ac)}_{X,Y}(x,\infty) + (1-p)F^{(si)}_{X,Y}(x,x), \qquad \text{and} \\ P(Y \le y) &= F_Y(y) = pF^{(ac)}_{X,Y}(\infty,y) + (1-p)F^{(si)}_{X,Y}(y,y). \end{aligned}$$

Therefore, it is clear that in general it will be a mixture distribution, but with the proper choice of p, $F^{(ac)}_{X,Y}(x,y)$ and $F^{(si)}_{X,Y}(x)$, it may not be a mixture distribution. Some of the properties can be easily obtained for (X,Y), such that $P(X=Y)=p$, $P(X \neq Y) = 1-p$, and

$$\begin{aligned} P(X<Y) &= p\int_0^\infty \int_x^\infty f^{(ac)}_{X,Y}(u,v)dvdu \\ P(X>Y) &= p\int_0^\infty \int_0^x f^{(ac)}_{X,Y}(u,v)dvdu. \end{aligned}$$

We can further obtain quite convenoently the distribution of $\max\{X,Y\}$ and $\min\{X,Y\}$, i.e.

$$\begin{aligned} P(\max\{X,Y\} \le x) &= pF^{(ac)}_{X,Y}(x,x) + (1-p)F^{(si)}_{X,Y}(x,x) \quad \text{and} \\ P(\min\{X,Y\} \ge x) &= pS^{(ac)}_{X,Y}(x,x) + (1-p)S^{(si)}_{X,Y}(x,x). \end{aligned}$$

This model can be used quite conveniently for modeling data from a dependent series system, dependent parallel system, analyzing dependent competing risks data and also dependent complementory risks data. We will explain those in detail in the subsequent sections.

4 Bivariate Weibull Distribution

4.1 Joint, Marginal and Conditional PDFs

The main aim of this section is to define a bivariate Weibull (BWE) distribution based on the proposed method, which has a close similarity with the popular MOWE distribution, but it is more flexible than the MOBW distribution. We discuss different properties of the BWE distribution and provide the estimation method. We will also explore how this model can be used to analyze dependent competing risks data.

Consider the bivariate distribution which has the following absolute continuous part and the singular part.

$$f^{(ac)}_{X,Y}(x,y) = c\begin{cases} f_{WE}(x;\alpha,\delta_1)f_{WE}(y;\alpha,\delta_2) & \text{if} \quad x < y \\ f_{WE}(x;\alpha,\delta_3)f_{WE}(y;\alpha,\delta_4) & \text{if} \quad y < x, \end{cases} \tag{4.1}$$

where $c^{-1} = \dfrac{\delta_1}{\delta_1+\delta_2} + \dfrac{\delta_4}{\delta_3+\delta_4}$, and

$$f^{(si)}_{X,Y}(x,y) = \begin{cases} f_{WE}(x;\alpha,\delta_5) & \text{if} \quad x = y \\ 0 & \text{if} \quad x \neq y. \end{cases} \tag{4.2}$$

Then the random variable (X,Y) has the following joint PDF of (X,Y):

$$f_{X,Y}(x,y) = pf^{(ac)}_{X,Y}(x,y) + (1-p)f^{(si)}_{X,Y}(x,y). \tag{4.3}$$

We make the following restrictions on the parameter.

$$\delta_1+\delta_2 = \delta_3+\delta_4 = \theta \quad \text{(say)} \qquad \text{and} \qquad \delta_5 = \delta_1+\delta_2 = \theta. \tag{4.4}$$

These restrictions have been made so that the proposed bivariate distribution has a similar structure as the MOBW distribution and at the same time it is more flexible than the later. From now on a bivariate distribution with the joint PDF (4.3) and with the restriction (4.4) will be called BWE distribution. It may be mentioned that $f^{(ac)}_{X,Y}(x,y)$ is continuous for all $0 < x,y < \infty$, when $\delta_1+\delta_2 = \delta_3+\delta_4$ and $|\delta_1-\delta_2|$ $= |\delta_3-\delta_4|$, similar to the MOBW distribiuion, otherwise it is not continuous on the line $x = y$. When $\alpha = 1$, we call it as the bivariate exponential (BEX) distribution.

The corresponding survival function, $S_{X,Y}(x,y)$ for $x \le y$ becomes:

$$S_{X,Y}(x,y) = pc\left\{(e^{-\delta_1 x^\alpha} - e^{-\delta_1 y^\alpha})e^{-\delta_2 y^\alpha} + \frac{\delta_1+\delta_4}{\theta}e^{-\theta y^\alpha}\right\} + (1-p)e^{-\theta y^\alpha} \tag{4.5}$$

and for $x > y$,

$$S_{X,Y}(x,y) = pc\left\{(e^{-\delta_4 y^\alpha} - e^{-\delta_4 x^\alpha})e^{-\delta_3 x^\alpha} + \frac{\delta_1+\delta_4}{\theta}e^{-\theta x^\alpha}\right\} + (1-p)e^{-\theta x^\alpha}. \tag{4.6}$$

It can be easily seen that if we take:

$$\delta_1 = \lambda_1, \quad \delta_2 = \lambda_2+\lambda_3, \quad \delta_3 = \lambda_1+\lambda_3, \quad \delta_4 = \lambda_2, \quad p = \frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2+\lambda_3},$$

it satisfies the constraint (4.4), and $S_{X,Y}(x,y)$ becomes

$$S_{X,Y}(x,y) = \begin{cases} e^{-\lambda_1 x^\alpha} e^{-(\lambda_2+\lambda_3) y^\alpha} & \text{if} \quad x \le y \\ e^{-(\lambda_1+\lambda_3) x^\alpha} e^{-\lambda_2 y^\alpha} & \text{if} \quad x > y. \end{cases}$$

It shows that the proposed BWE distribution becomes the MOBW distribution, and when $\alpha = 1$, it reduces to the MOBE distribution.

It may be noted that the proposed BWE distribution is more flexible than the MOBW distribution, as the former has one extra parameter. Moreover, it is well known that the estimation of the unknown parameters in case of MOBW distribution is not a trivial issue, see for example Kundu and Dey (2009), where as it is observed that in case of BWE distribution, the maximum likelihood estimators can be obtained in a routine manner. Hence, the proposed BWE distribution can be used quite effectively for data analysis purposes.

Now we consider the marginal distribution functions of BWE.

$$\begin{aligned} S_X(x) &= P(X > x) = p\frac{\delta_1+\delta_4}{\theta} e^{-\delta_3 x^\alpha} + \left(1 - p\frac{\delta_1+\delta_4}{\theta}\right) e^{-\theta x^\alpha} \\ S_Y(y) &= P(Y > y) = p\frac{\delta_1+\delta_4}{\theta} e^{-\delta_2 y^\alpha} + \left(1 - p\frac{\delta_1+\delta_4}{\theta}\right) e^{-\theta y^\alpha}. \end{aligned}$$

It is interesting to see that if $p\dfrac{\delta_1+\delta_4}{\delta_1+\delta_2} < 1$, then the marginal distribution functions can be written as the mixture of two Weibull distribution functions, and if $p\dfrac{\delta_1+\delta_4}{\delta_1+\delta_2} > 1$, then it can be written as the generalized mixture of Weibull distributions, see for example Franco et al. (2014). The PDFs of X and Y can be written as

$$\begin{aligned} f_X(x) &= p\frac{\delta_1+\delta_4}{\theta} f_{WE}(x;\alpha,\delta_3) + \left(1 - p\frac{\delta_1+\delta_4}{\theta}\right) f_{WE}(x;\alpha,\theta) \\ f_Y(y) &= p\frac{\delta_1+\delta_4}{\theta} f_{WE}(y;\alpha,\delta_2) + \left(1 - p\frac{\delta_1+\delta_4}{\theta}\right) f_{WE}(y;\alpha,\theta), \end{aligned}$$

respectively. Since the marginals are mixtures of Weibull distributions the PDFs can take variety of shapes. It can be increasing, decreasing and even bimodal also. Moreover, the hazard functions of the marginals also can be of different types. The PDFs and hazard functions of the marginals for different parameter values have been plotted in Figure 1. It may be mentioned that Figures 1(a)-1(c) correspond to mixture of Weibull distributions where as Figure 1(d) corresonds to generalized mixture of Weibull distributions.

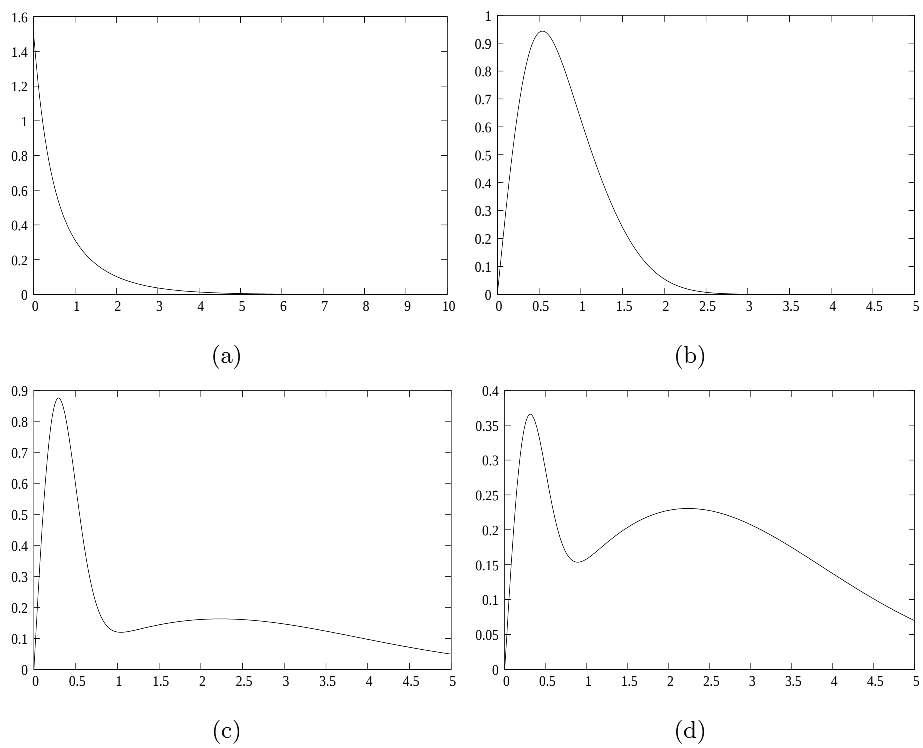

Figure 1: The PDF plot of the marginal distribution of X, when (a) $\alpha = 1.0$, $\delta_1 = 2.0$, $\delta_2 = 1.0$, $\delta_3 = 1.0$, $\delta_4 = 2.0$, $p = 0.75$, (b) $\alpha = 2.0$, $\delta_1 = 2.0$, $\delta_2 = 1.0$, $\delta_3 = 1.0$, $\delta_4 = 2.0$, $p = 0.75$, (c) $\alpha = 2$, $\delta_1 = 3.0$, $\delta_2 = 3.0$, $\delta_3 = 0.1$, $\delta_4 = 5.9$, $p = 0.6$, (d) $\alpha = 2$, $\delta_1 = 3.0$, $\delta_2 = 3.0$, $\delta_3 = 0.1$, $\delta_4 = 5.9$, $p = 0.85$.

In Figure 2 we provide the hazard functions of the marginal distribution X for different parameter values. It is clear that it can take variety of shapes. In this case also Figures 2(a) - 2(c) corresond to the mixture of Weibull distributions, and Figure 2(d) corresonds to the generalized mixture of Weibull distributions.

Now we discuss some conditional distributions which will be of interest in data analysis, and it may have some independent interests also. For example, if (X, Y)

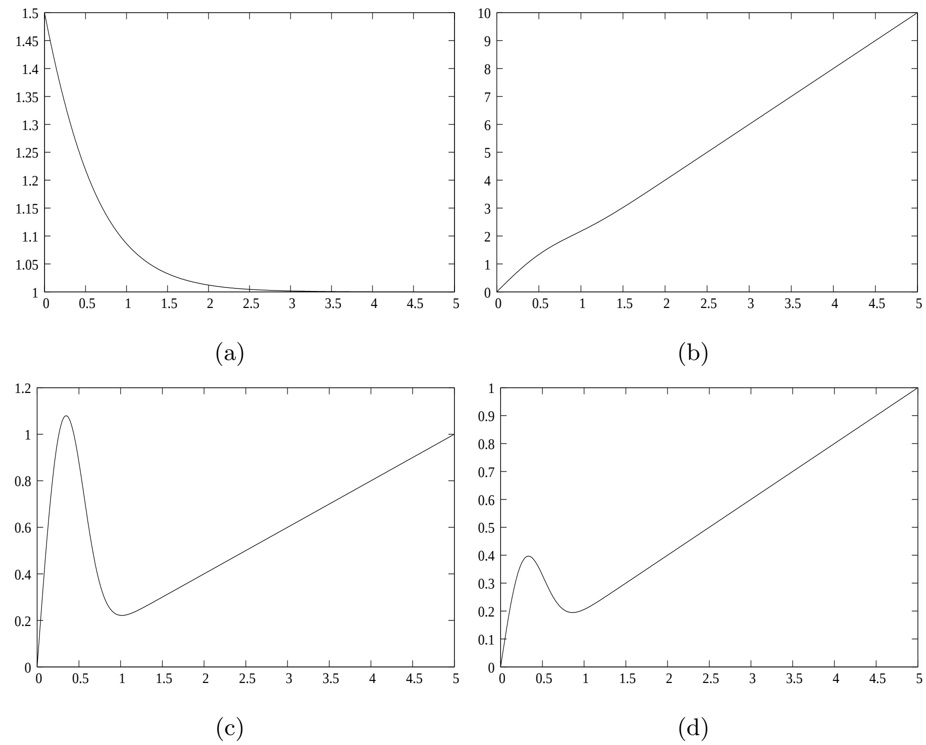

Figure 2: The hazard function plot of the marginal distribution of X, when (a) α = 1.0, δ_1 = 2.0, δ_2 = 1.0, δ_3 = 1.0, δ_4 = 2.0, p = 0.75, (b) α = 2.0, δ_1 = 2.0, δ_2 = 1.0, δ_3 = 1.0, δ_4 = 2.0, p = 0.75, (c) α = 2, δ_1 = 3.0, δ_2 = 3.0, δ_3 = 0.1, δ_4 = 5.9, p = 0.6, (d) α = 2, δ_1 = 3.0, δ_2 = 3.0, δ_3 = 0.1, δ_4 = 5.9, p = 0.85.

has a BWE as described in (4.3) then it can be easily seen that

$$X|\{X < Y\} \sim \text{WE}(\alpha, \theta) \quad \text{and} \quad Y|\{Y < X\} \sim \text{WE}(\alpha, \theta). \tag{4.7}$$

Moreover, the conditional PDF of $Y|\{X < Y\}$ and $X|\{Y < X\}$ can be written as

$$f_{Y|\{X<Y\}}(y) = \frac{\theta}{\delta_1}\alpha\delta_2 y^{\alpha-1}e^{-\delta_2 y^\alpha}(1 - e^{-\delta_1 y^\alpha}) \tag{4.8}$$

$$f_{X|\{Y<X\}}(y) = \frac{\theta}{\delta_4}\alpha\delta_3 y^{\alpha-1}e^{-\delta_3 y^\alpha}(1 - e^{-\delta_4 y^\alpha}). \tag{4.9}$$

It can be seen that (4.8) and (4.9) are the PDFs of the weighted Weibull (WWE) distribution, as introduced by Gupta and Kundu (2009), see also Shahbaz, Shahbaz and Butt (2010) in this respect. It may be mentioned that the WWE distribution has an interesting interpretation similar to the skew normal distribution as introduced by the Azzalini (1985). For several interesting properties on WWE distribution, one may refer to Al-Mutairi, Ghitany and Kundu (2018).

4.2 Absolute Continuous Part of a BWE and its Generation

In this section we study some basic feature of the absolute continuous part of the proosed BWE distribution. Suppose (X, Y) has a BWE distribution with the absolute continuous part as given in (4.1). Let us assume that an absolute continuous bivariate random variable (U, V) has the joint PDF

$$f_{U,V}(u, v) = \frac{\theta}{\delta_1 + \delta_4} \begin{cases} f_{WE}(u; \alpha, \delta_1) f_{WE}(v; \alpha, \delta_2) & \text{if} \quad u < v \\ f_{WE}(u; \alpha, \delta_3) f_{WE}(v; \alpha, \delta_4) & \text{if} \quad v < u. \end{cases} \tag{4.10}$$

It may be mentioned that the joint PDF of (U, V) may be compared with the joint PDF of the Block and Basu bivariate Weibull (BBBW) distribution. The BBBW distribution can be obtanied from a MOBW distribution by removing the singular component, see for example the original article by Block and Basu (1974), see also Pradhan and Kundu (2016) in this respect. It may be recalled that joint PDF of a BBBW distribution with parameters $\alpha, \lambda_0, \lambda_1, \lambda_2$, can be written as follows:

$$f_{BBBW}(u, v) = \frac{\lambda_0 + \lambda_1 + \lambda_2}{\lambda_1 + \lambda_2} \begin{cases} f_{WE}(u; \alpha, \lambda_1) f_{WE}(v; \alpha, \lambda_0 + \lambda_2) & \text{if} \quad u < v \\ f_{WE}(v; \alpha, \lambda_0 + \lambda_1) f_{WE}(v; \alpha, \lambda_2) & \text{if} \quad v < u. \end{cases} \tag{4.11}$$

It is clear that the joint PDF (4.11) can be obtained as a special case of (4.10). Now we provide the shape of the joint PDF of (4.10). It may be noted that if $\alpha \leq 1$, then the joint PDF of (U, V) is a decreasing function both in u and v directions for all values of δ_1, δ_2, δ_3 and δ_4. Therefore, we are mainly interested when $\alpha > 1$ and then we have the following result.

THEOREM 1: For $\alpha > 1$, if (a) $\delta_2 < \delta_1$ and $\delta_4 > \delta_3$, then it is a bimodal function and the two modes are at (i) $\left(\left[\frac{\alpha - 1}{\alpha\delta_1}\right]^{1/\alpha}, \left[\frac{\alpha - 1}{\alpha\delta_2}\right]^{1/\alpha} \right)$ and (ii) $\left(\left[\frac{\alpha - 1}{\alpha\delta_3}\right]^{1/\alpha}, \left[\frac{\alpha - 1}{\alpha\delta_4}\right]^{1/\alpha} \right)$,
(b) $\delta_1 < \delta_2$, $\delta_4 < \delta_3$, then it is unimodal, and the mode is at $\left(\left[\frac{2(\alpha - 1)}{\alpha(\delta_1 + \delta_2)}\right]^{1/\alpha}, \left[\frac{2(\alpha - 1)}{\alpha(\delta_1 + \delta_2)}\right]^{1/\alpha} \right)$,

(c) $\delta_1 < \delta_2$, $\delta_4 > \delta_3$ and $\delta_1\delta_2 > \delta_3\delta_4$, then it is a bimodal function, and and the two modes are at (i) $\left(\left[\frac{2(\alpha-1)}{\alpha(\delta_1+\delta_2)}\right]^{1/\alpha}, \left[\frac{2(\alpha-1)}{\alpha(\delta_1+\delta_2)}\right]^{1/\alpha}\right)$ and (ii) $\left(\left[\frac{\alpha-1}{\alpha\delta_3}\right]^{1/\alpha}, \left[\frac{\alpha-1}{\alpha\delta_4}\right]^{1/\alpha}\right)$,

(d) $\delta_1 < \delta_2$, $\delta_4 > \delta_3$ and $\delta_1\delta_2 < \delta_3\delta_4$, then it is unimodal and the mode is at $\left(\left[\frac{\alpha-1}{\alpha\delta_3}\right]^{1/\alpha}, \left[\frac{\alpha-1}{\alpha\delta_4}\right]^{1/\alpha}\right)$.

Proof: The proof is not very difficult to obtain. It can be obtained by studying the derivatives of the log of the joint PDF of the BWE distribution. The details are avoided. □

The following Figure 3 provides the contour plots of the absolute continuous part of the BWE distribution for different parameter values.

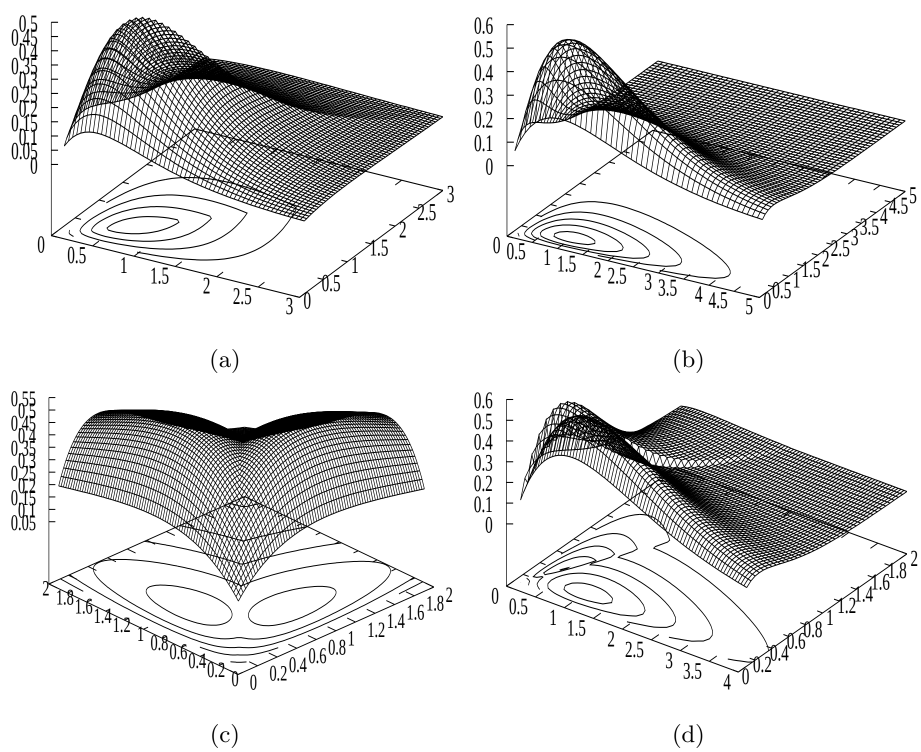

Figure 3: The contour plot of the joint PDF of the absolute continuous part of the BWE distribution when (a)$\alpha = 2$, $\delta_1 = 1.0$, $\delta_2 = 2.0$, $\delta_3 = 2.0$, $\delta_4 = 1.0$, (b) $\alpha = 2$, $\delta_1 = 1.0$, $\delta_2 = 2.0$, $\delta_3 = 1.0$, $\delta_4 = 2.0$, (c) $\alpha = 2$, $\delta_1 = 2.0$, $\delta_2 = 1.0$, $\delta_3 = 1.0$, $\delta_4 = 2.0$, (d) $\alpha = 2$, $\delta_1 = 2.0$, $\delta_2 = 3.0$, $\delta_3 = 1.0$, $\delta_4 = 4.0$.

It can be easily checked that if (U,V) has the joint PDF (4.10), then $P(U < V) = \dfrac{\delta_1}{\delta_1+\delta_4}$ and $P(U > V) = \dfrac{\delta_4}{\delta_1+\delta_4}$. The following decomposition is useful to generate samples from (U,V), they may have some independent interests also. The joint PDF of (U,V) can be written as follows:

$$f_{U,V}(u,v) = \frac{\delta_1}{\delta_1+\delta_4} f_{U_1,V_1}(u,v) + \frac{\delta_4}{\delta_1+\delta_4} f_{U_2,V_2}(u,v). \tag{4.12}$$

Here

$$f_{U_1,V_1}(u,v) = \begin{cases} \frac{\delta_1+\delta_2}{\delta_1} f_{WE}(u;\alpha,\delta_1) f_{WE}(v;\alpha,\delta_2) & \text{if} \quad u < v \\ 0 & \text{if} \quad u \geq v \end{cases} \tag{4.13}$$

and

$$f_{U_2,V_2}(u,v) = \begin{cases} \frac{\delta_3+\delta_4}{\delta_4} f_{WE}(u;\alpha,\delta_3) f_{WE}(v;\alpha,\delta_4) & \text{if} \quad u > v \\ 0 & \text{if} \quad u \leq v. \end{cases} \tag{4.14}$$

It follows that if (U_1,V_1) has a joint PDF (4.13), then $U_1 \sim \text{WE}(\alpha,\delta_1+\delta_2)$ and $P(V_1 > v | U_1 = u) = e^{-\delta_2(v^\alpha - u^\alpha)}$, for $v > u$. Similarly, if (U_2,V_2) has a joint PDF (4.14), then $V_2 \sim \text{WE}(\alpha,\delta_3+\delta_4)$ and $P(U_2 > u | V_2 = v) = e^{-\delta_3(u^\alpha - v^\alpha)}$, for $u > v$. It is quite simple to generate random samples from (U_1,V_1) and (U_2,V_2), and hence, generation random samples from (U,V) is straight forward.

Note that if (X,Y) has BWE distribution, then it can be written as follows:

$$(X,Y) = \begin{cases} (U_1,V_1) & \text{with probability} \quad \frac{p\delta_1}{\delta_1+\delta_4} \\ (U_2,V_2) & \text{with probability} \quad \frac{p\delta_4}{\delta_1+\delta_4} \\ (W,W) & \text{with probability} \quad 1-p, \end{cases} \tag{4.15}$$

here (U_1,V_1) and (U_2,V_2) are same as defined above, and $W \sim \text{WE}(\alpha,\delta_1+\delta_2)$. The above decomposition (4.15) can be used quite effectively to generate random samples from a BWE distribution.

We have provided the scatter plots of (X,Y) generated from BWE distribution for different parameter values in Figure 4. In each case we have reported the corresponding sampling correlation (r) also based 100 data points. It may be observed that the sample correlation coefficient can be negative also in this case for certain set of parameters.

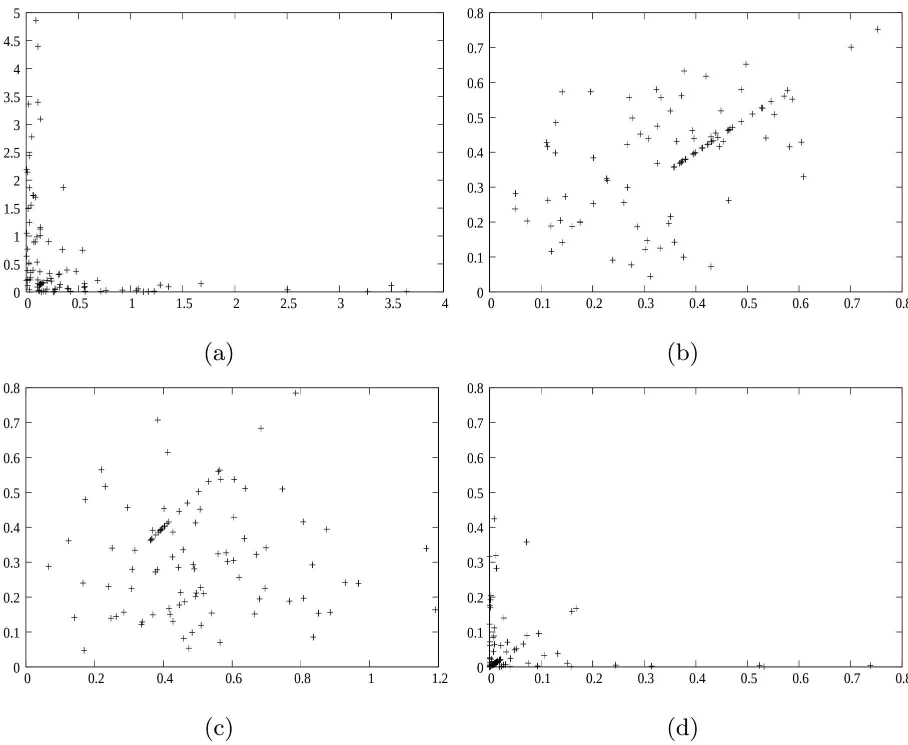

Figure 4: The scatter of (X, Y) generated from BWE distribution when (a) α = 1.0, δ_1 = 10.0, δ_2 = 1.0, δ_3 = 1.0, δ_4 = 10.0, p = 0.75, r = -0.32 (b) α = 2.0, δ_1 = 1.0, δ_2 = 10.0, δ_3 = 10.0, δ_4 = 1.0, p = 0.75, r = 0.71 (c) α = 2, δ_1 = 1.0, δ_2 = 10.0, δ_3 = 5.0, δ_4 = 6.0, p = 0.75, r = -0.28 (d) α = 0.5, δ_1 = 8.0, δ_2 = 3.0, δ_3 = 5.0, δ_4 = 6.0, p = 0.5, r = 0.009.

4.3 Maximum Likelihood Estimators

In this section we disucss about the maximum likelihood estimators of the unknown parameters. It is assumed we have a random sample of size n from a BWE distribution with the constraint on the parameters (4.4). Therefore, the proposed BWE distribution has five independent parameters. Let $\mathcal{D} = \{(x_1, y_1), \ldots, (x_n, y_n)\}$ be a random sample of size n from a BWE distribution, and we use the following notation

$$I_1 = \{i : x_i < y_i\}, \quad I_2 = \{i : x_i > y_i\} \quad I_0 = \{i : x_i = y_i = u_i\}.$$

and $|I_1| = n_1$, $|I_2| = n_2$, $|I_0| = n_0$. Moreover, we use $\theta = \delta_1 + \delta_2 = \delta_3 + \delta_4$, as before. Therefore, it is assumed that p, α, θ, δ_2, δ_3 are the independent parameters of the proposed model, and $\mathbf{\Theta} = (p, \alpha, \mathbf{\Gamma})^\top$, where $\mathbf{\Gamma} = (\theta, \delta_2, \delta_3)^\top$. Now based on the observed sample $\mathcal{D}$, the log-likelihood function can be written as

$$\begin{aligned} l(p, \alpha, \mathbf{\Gamma}|\mathcal{D}) \quad = \quad & (n_1 + n_2) \{\ln p - \ln(2\theta - \delta_2 - \delta_3) + \ln \theta\} + 2n_1 \ln \alpha + n_1 \ln \delta_2 + \\ & n_1 \ln(\theta - \delta_2) + (\alpha - 1) \sum_{i \in I_1 \cup I_2} \ln x_i - (\theta - \delta_2) \sum_{i \in I_1} x_i^\alpha + \\ & (\alpha - 1) \sum_{i \in I_1 \cup I_2} \ln y_i - \delta_2 \sum_{i \in I_1} y_i^\alpha + 2n_2 \ln \alpha + n_2 \ln \delta_3 + n_2 \ln(\theta - \delta_3) - \\ & \delta_3 \sum_{i \in I_2} x_i^\alpha - (\theta - \delta_3) \sum_{i \in I_2} y_i^\alpha + n_0 \ln(1 - p) + n_0 \ln \alpha + n_0 \ln \theta + \\ & (\alpha - 1) \sum_{i \in I_0} \ln u_i - \theta \sum_{i \in I_0} u_i^\alpha. \end{aligned} \tag{4.16}$$

The maximum likelihood estimators (MLEs) of $p, \alpha, \mathbf{\Gamma}$ can be obtained by maximizing $l(p, \alpha, \mathbf{\Gamma}|\mathcal{D})$ with respect to the unknown parameters. It can be easily seen that the MLE of p becomes

$$\widehat{p} = \frac{n_1 + n_2}{n},$$

and the MLEs of $(\alpha, \mathbf{\Gamma})^\top$ can be obtained by maximizing

$$\begin{aligned} l_0(\alpha, \mathbf{\Gamma}) \quad = \quad & (n_1 + n_2) \{-\ln(2\theta - \delta_2 - \delta_3) + \ln \theta\} + 2n_1 \ln \alpha + n_1 \ln \delta_2 + n_1 \ln(\theta - \delta_2) + \\ & \alpha \sum_{i \in I_1 \cup I_2} \ln x_i - (\theta - \delta_2) \sum_{i \in I_1} x_i^\alpha + \alpha \sum_{i \in I_1 \cup I_2} \ln y_i - \delta_2 \sum_{i \in I_1} y_i^\alpha + \\ & 2n_2 \ln \alpha + n_2 \ln \delta_3 + n_2 \ln(\theta - \delta_3) - \delta_3 \sum_{i \in I_2} x_i^\alpha - (\theta - \delta_3) \sum_{i \in I_2} y_i^\alpha + \\ & n_0 \ln \alpha + n_0 \ln \theta + \alpha \sum_{i \in I_0} \ln u_i - \theta \sum_{i \in I_0} u_i^\alpha, \end{aligned} \tag{4.17}$$

with respect to the unknown parameters. It is a four dimensional optimization problem. If we try to solve directly, then we can obtain the normal equations and we need to solve four non-linear equations simultaneously. To avoid that we have used the profile likelihood method, and the method provided by Song, Fan and Kalbfleisch (2005). It is observed that the the MLEs of the unknown parameters can be obtained by solving only one non-linear equation. The details are provided in Appendix A. Once we obtain the MLEs of the unknown parameters, the observed Fisher information matrix can be easily constructed, and the asymptotic confidence intervals of the unknown parameters also can be computed.

It may be noted that the algorithm which has been described in the previous section, is an iterative process, hence good initial estimates are needed for α, θ, δ_2 and δ_3. Now we describe how to obtain the initial guesses.

Now to compute α and θ, we use the results (4.7). The data points which are obtained as below

$$\{x_i : i \in I_0 \cup I_1\}, \quad \text{and} \quad \{y_i : i \in I_2\},$$

are i.i.d. WE(α, θ) random variables. Hence, the estimates of α and θ can be obtained quite conveniently as several efficient methods are available to estimate the shape and scale parameters of a Weibull distribution. Further, the initial estimates of δ_1 and δ_4 can be obtained from

$$\{x_i : i \in I_2\} \quad \text{and} \quad \{y_i : i \in I_1\},$$

and using the results (4.8) and (4.9). Note that $\{x_i : i \in I_2\}$ and $\{y_i : i \in I_1\}$, are WWE distributions. Hence, assuming the shape parameter α to be known, the method proposed by Gupta and Kundu (2009) can be used quite conveniently to estimate δ_1 and δ_4.

5 Data Analysis

In this section we perform the analyses of two data sets, one simulated and one real data set.

5.1 Simulated Data Set:

We have generated a data set with $n = 50$, and $p = 0.75$, $\alpha = 1.5$, $\delta_1 = \delta_2 = \delta_3 = \delta_4$ = 1.0. The data set has been plotted in Figure 5 In this case the intitial estimates of α, θ, δ_1 and δ_4 can be obtained as

$$\widetilde{\alpha} = 1.142, \quad \widetilde{\theta} = 1.842, \quad \widetilde{\delta}_1 = 0.597, \quad \widetilde{\delta}_4 = 0.911.$$

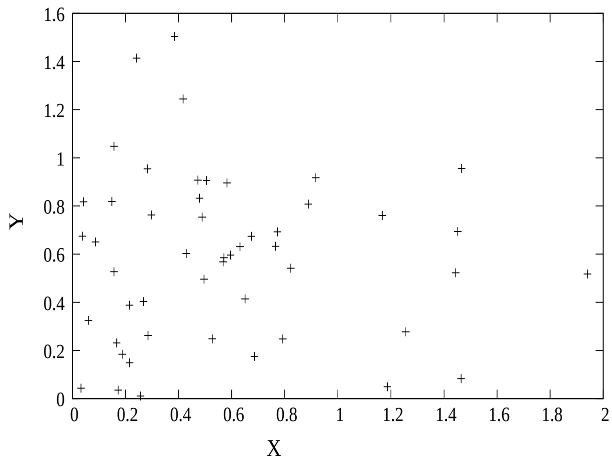

Figure 5: The scatter of (X, Y) for simulated sample.

We have used the above initial estimates to compute the MLEs of the unknown parameters. We have reported the MLEs and the associated 95% confidence intervals (in brackets) based on the observed Fisher information matrix.

$$\widehat{\alpha} = 1.368(\mp 0.352), \quad \widehat{\theta} = 1.833(\mp 0.451), \quad \widehat{\delta}_1 = 0.864(0.235), \quad \widehat{\delta}_2 = 0.969(\mp 0.268),$$

$$\widehat{\delta}_3 = 0.877(\mp 0.235), \quad \widehat{\delta}_4 = 0.956(0.254), \quad \widehat{p} = 0.88(0.17).$$

5.2 Soccer Data

In this section we have analyzed on soccer data set based on the proposed BWE model. The data set has been obtained from Meintanis (2007) and it represents the soccer data where at least one goal has been scored by the home team and at least one goal has been scored directly either from a penalty kick, foul kick or any other direct kick. All of them together we call tham as the *kick goal*. Here X reresents the time in minutes of the first *kick goal* by any team and Y represents the time in minutes of the first goal of any type scored by the home team. It may be noted that in this case all possibilities are open, namely $X < Y$, $X > Y$ or $X = Y$. The data set has been presented in Table 1.

Before procedding further, we have divided all the data points by 100 mainly

Table 1: UEFA Champion's League data

2005-2006	X	Y	2004-2005	X	Y
Lyon-Real Madrid	26	20	Internazionale-Bremen	34	34
Milan-Fenerbahce	63	18	Real Madrid-Roma	53	39
Chelsea-Anderlecht	19	19	Man. United-Fenerbahce	54	7
Club Brugge-Juventus	66	85	Bayern-Ajax	51	28
Fenerbahce-PSV	40	40	Moscow-PSG	76	64
Internazionale-Rangers	49	49	Barcelona-Shakhtar	64	15
Panathinaikos-Bremen	8	8	Leverkusen-Roma	26	48
Ajax-Arsenal	69	71	Arsenal-Panathinaikos	16	16
Man. United-Benfica	39	39	Dynamo Kyiv-Real Madrid	44	13
Real Madrid-Rosenborg	82	48	Man. United-Sparta	25	14
Villarreal-Benfica	72	72	Bayern-M. TelAviv	55	11
Juventus-Bayern	66	62	Bremen-Internazionale	49	49
Club Brugge-Rapid	25	9	Anderlecht-Valencia	24	24
Olympiacos-Lyon	41	3	Panathinaikos-PSV	44	30
Internazionale-Porto	16	75	Arsenal-Rosenborg	42	3
Schalke-PSV	18	18	Liverpool-Olympiacos	27	47
Barcelona-Bremen	22	14	M. Tel-Aviv-Juventus	28	28
Milan-Schalke	42	42	Bremen-Panathinaikos	2	2
Rapid-Juventus	36	52			

for numerical purposes. It is not going to make any difference in the inference procedure. It is observed that in this case $N_1 = 6$, $N_2 = 17$ and $N_0 = 14$. We obtain the initial estimates of α, θ, δ_1 and δ_4 as

$$\widetilde{\alpha} = 1.477, \quad \widetilde{\theta} = 5.341, \quad \widetilde{\delta}_1 = 4.199, \quad \widetilde{\delta}_4 = 4.699.$$

Based on these initial estimates we compute the MLEs of the unknown parameters. The MLEs and the associated 95% confidence intervals (in brackets) based on the observed Fisher information matrix are reported below.

$$\widehat{\alpha} = 1.649(\mp 0.473), \quad \widehat{\theta} = 5.981(\mp 1.114), \quad \widehat{\delta}_1 = 0.483(0.097), \quad \widehat{\delta}_2 = 5.499(\mp 1.654),$$

$$\widehat{\delta}_3 = 4.653(\mp 0.998), \quad \widehat{\delta}_4 = 1.323(0.189), \quad \widehat{p} = 0.622(0.201).$$

6 Conclusions

In this paper we have proposed a general construncation of a bivariate distribution with a singular component. It is well known that most of the existing methods which can be used to generate a bivariate distribution with a singular component can be classifed into two classes, namely the minimization approach proposed by Marshall and Olkin (1967) and the maximization approach proposed by Kundu and Gupta (2009). The present method actually unifies both the approaches. All the existing bivariate distributions with a singular component can be obtained as special cases of the proposed model. We have considered one specific model based on the Weibull distributions and discussed some of its properties, and showed how it can be used in practice. It may be mentioned that this model can be used quite effectively as a dependent competing risks model and it can be extended to the multivariate case also. More work is needed along that direction.

Acknowledgements: The author would like to thank the reviewers for their helpful comments which have helped to imporve the manuscript.

Appendix A

In this section we show how we can maximize $l_0(\alpha, \mathbf{\Gamma})$ with respect to the unknown parameters. This is a four dimensional optimization problem. To avoid that we use the profile likelihood method. In that case for a fixed α first we maximize with respect to $\mathbf{\Gamma}$ and then we maximize with respect to α. Now maximizing $l_0(\alpha, \mathbf{\Gamma})$ with respect to $\mathbf{\Gamma}$ for a fixed α is a three dimensional optimization problem, and we use the method of Song, Fan, and Kalbfleisch (2005) to optimize it. It can be described as follows. Let us write

$$l_0(\alpha, \mathbf{\Gamma}) = l_1(\mathbf{\Gamma}) + l_2(\mathbf{\Gamma}) + l_3(\alpha),$$

here

$$\begin{aligned}
l_3(\alpha) &= \ln \alpha (2n_1 + 2n_2 + n_0) + \alpha \left(\sum_{i \in I_1 \cup I_2} \ln x_i + \sum_{i \in I_1 \cup I_2} \ln y_i + \sum_{i \in I_0} \ln u_i \right), \\
l_{1\alpha}(\mathbf{\Gamma}) &= n \ln \theta - \theta \left(\sum_{i \in I_1} x_i^\alpha + \sum_{i \in I_2} y_i^\alpha + \sum_{i \in I_0} u_i^\alpha \right) + n_1 \ln \delta_2 - \delta_2 \sum_{i \in I_1} (y_i^\alpha - x_i^\alpha) \\
&\quad + n_2 \ln \delta_3 - \delta_3 \sum_{i \in I_2} (x_i^\alpha - y_i^\alpha) \\
l_{2\alpha}(\mathbf{\Gamma}) &= -(n_1 + n_2) \ln(2\theta - \delta_2 - \delta_3) + n_1 \ln(\theta - \delta_2) + n_2 \ln(\theta - \delta_3).
\end{aligned}$$

Therefore, for a given α, if $\widehat{\mathbf{\Gamma}}_\alpha$ maximizes $l_1(\mathbf{\Gamma}) + l_2(\mathbf{\Gamma})$, then the MLE of α, say $\widehat{\alpha}$ can be obtained by the argument maximum of $l_0(\alpha, \widehat{\mathbf{\Gamma}}_\alpha)$, and the MLE of $\mathbf{\Gamma}$, say $\widehat{\mathbf{\Gamma}}$ can be obtained as $\widehat{\mathbf{\Gamma}} = \widehat{\mathbf{\Gamma}}_{\widehat{\alpha}}$

For a given α, we want to maximize $l_{1\alpha}(\mathbf{\Gamma}) + l_{2\alpha}(\mathbf{\Gamma})$, with respect to $\mathbf{\Gamma}$. It means we need to find the solution of the vector equation:

$$\dot{l}_{1\alpha}(\mathbf{\Gamma}) + \dot{l}_{2\alpha}(\mathbf{\Gamma}) = \mathbf{0},$$

equivalently

$$\dot{l}_{1\alpha}(\mathbf{\Gamma}) = -\dot{l}_{2\alpha}(\mathbf{\Gamma}),$$

here $\mathbf{0} = (0,0,0)^\top$,

$$\dot{l}_{1\alpha}(\mathbf{\Gamma}) = \left(\frac{\partial}{\partial \theta} l_{1\alpha}(\mathbf{\Gamma}), \frac{\partial}{\partial \delta_2} l_{1\alpha}(\mathbf{\Gamma}), \frac{\partial}{\partial \delta_3} l_{1\alpha}(\mathbf{\Gamma}) \right)^\top, \dot{l}_{2\alpha}(\mathbf{\Gamma}) = \left(\frac{\partial}{\partial \theta} l_{2\alpha}(\mathbf{\Gamma}), \frac{\partial}{\partial \delta_2} l_{2\alpha}(\mathbf{\Gamma}), \frac{\partial}{\partial \delta_3} l_{2\alpha}(\mathbf{\Gamma}) \right)^\top.$$

Song, Fan, and Kalbfleisch (2005) suggested the following, first solve

$$\dot{l}_{1\alpha}(\mathbf{\Gamma}) = 0, \tag{6.1}$$

if $\mathbf{\Gamma}^{(0)}$ is the solution of (6.1), then find $\mathbf{\Gamma}^{(1)}$ such that

$$\dot{l}_{1\alpha}(\mathbf{\Gamma}) = -\dot{l}_{2\alpha}(\mathbf{\Gamma}^{(0)}).$$

Continue the process, until the convergence takes place. We will provide the explicit expressions of If $\mathbf{\Gamma}^{(j)} = (\theta^{(j)}, \delta_2^{(j)}, \delta_3^{(j)})^\top$, the value of $\mathbf{\Gamma}^{(j)}$ at the j-th iteration. Let us use the following notations:

$$A(\alpha) = \sum_{i \in I_1} x_i^\alpha + \sum_{i \in I_2} y_i^\alpha + \sum_{i \in I_0} u_i^\alpha, \quad B(\alpha) = \sum_{i \in I_1} (y_i^\alpha - x_i^\alpha), \quad C(\alpha) = \sum_{i \in I_2} (x_i^\alpha - y_i^\alpha),$$

$a_0 = b_0 = c_0 = 0$, and for $j = 1, 2, \ldots$,

$$\begin{aligned}
a_j &= \frac{n_1 + n_2}{2\theta^{(j)} - \delta_2^{(j)} - \delta_3^{(j)}} - \frac{n_1}{\theta^{(j)} - \delta_2^{(j)}} - \frac{n_2}{\theta^{(j)} - \delta_3^{(j)}} \\
b_j &= -\frac{n_1 + n_2}{2\theta^{(j)} - \delta_2^{(j)} - \delta_3^{(j)}} + \frac{n_1}{\theta^{(j)} - \delta_2^{(j)}} \\
c_j &= -\frac{n_1 + n_2}{2\theta^{(j)} - \delta_2^{(j)} - \delta_3^{(j)}} + \frac{n_2}{\theta^{(j)} - \delta_3^{(j)}}.
\end{aligned}$$

Then

$$\theta^{(j+1)} = \frac{n}{A(\alpha) + a_j}, \quad \delta_2^{(j+1)} = \frac{n_1}{B(\alpha) + b_j}, \delta_3^{(j+1)} = \frac{n_2}{C(\alpha) + c_j}.$$

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Generalized Lehmann Alternative Type II Family of Distributions and Their Applications

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ABSTRACT

A new generalized family called Generalized Lehmann Alternative Type II (GLA2) family is introduced and studied in this paper. Special cases of this family using Uniform and Kumaraswamy distributions as base are developed and their statistical properties studied. Generalized Lehmann Alternative Type II Exponential (GLA2E) distribution is also developed and its statistical properties are obtained along with application. The new distribution is applied to a real data set to show the effectiveness of the distribution and it is verified that the new model is a better model than the existing exponential model and Marshall-Olkin extended exponential model. A detailed study on the record value theory associated with GLA2E distribution is conducted. Using the mean, variance and covariance of upper record values of the extended model, BLUE's of location and scale parameters are obtained and future records are predicted which has a number of practical uses. The 95% confidence interval for location and scale parameters are also computed. The result is applied to a real data set to validate the results. Entropy of record values is derived. This result will be

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useful in characterization of record values based on entropies and a quantification of information contained in each additional record value based on entropy measure.

Key words and Phrases: *Lehmann Alternative , Entropy, Hazard rate function, Kumaraswamy distribution, Marshall-Olkin distribution, Record value.*

1 Introduction

The properties and estimation methods for parameters of the exponentiated family of distributions have been studied by many authors, see Gupta and Kundu (2001a, 2001b, 2007), Pal et al. (2006), Nadarajah and Kotz (2006a) and Nadarajah et al. (2013). Tahir and Nadarajah (2015) discussed about Lehmann alternative type family of distributions. In the literature there exist two types of Lehmann alternative type family of distributions for obtaining the exponentiated family of distributions.

1.1 Lehmann Alternative 1 (LA1)

If F(x) is the cdf of the baseline distribution, then LA1 family of distributions is obtained by taking the β^{th} - power of F(x) so that

$$G(x) = (F(x))^{\beta}, \tag{1}$$

where $\beta > 0$ is a positive real parameter.The probability density function (pdf) corresponding to (1) is

$$g(x) = \beta f(x)(F(x))^{\beta-1}, \tag{2}$$

where $f(x) = \frac{d}{dx}F(x)$ denotes the pdf of F. For any lifetime random variable t, the survival (reliability) function (sf), $\overline{G}(t)$, the hazard (failure) rate function (hrf), h(t), the reversed hazard rate function (rhrf), r(t), and the cumulative hazard rate function (chrf), H(t), associated with (1) and (2) are $\overline{G}(t) = 1 - [F(t)]^{\beta}$, $h(t) = \beta f(t)[F(t)]^{\beta-1}[1-[F(t)]^{\beta}]^{-1}$, $r(t) = \beta f(t)[F(t)]^{-1}$ and $H(t) = -\log[1-[F(t)]^{\beta}]$.

1.2 Lehmann Alternative 2 (LA2)

If F(x) is the cdf and $\overline{F}(x) = 1 - F(x)$ is the sf of the baseline distribution, then the survival function of LA2 family of distributions is obtained by taking the β^{th} - power of $\overline{F}(x)$ so that

$$\overline{G}(x) = [\overline{F}(x)]^{\beta}, \tag{3}$$

where β is a positive real parameter. The LA2 cdf may also be written as

$$G(x) = 1 - [1 - F(x)]^{\beta}. \tag{4}$$

The pdf corresponding to (4) is

$$g(x) = \beta f(x)[1 - F(x)]^{\beta - 1} \tag{5}$$

For any lifetime random variable t, the sf, hrf, rhrf and chrf associated with (3) and (4) are $\overline{G}(t) = [1 - F(t)]^{\beta}$, $h(t) = \beta f(t)[1 - F(t)]^{-1}$, $r(t) = \beta f(t)[1 - F(t)]^{\beta-1}\{1 - [1 - F(t)]\beta\}^{-1}$ and $H(t) = -\beta \log[1 - F(t)]$.

Nadarajah and Kotz (2006 a), Nadarajah (2006) and Rao et al. (2013) used the LA2 approach for introducing exponentiated Fr*é*chet, exponentiated Gumbel and exponentiated log-logistic distributions. For more applications see Abd-Elfattah and Omima (2009), Abd-Elfattah et al. (2010), Rao et al. (2012, 2013), and Al-Nasser and Al-Omari (2013).

1.3 Marshall - Olkin Family of Distributions

Let X be a rv with a distribution function $F(x)$ and survival function $\overline{F}(x)$. By adding a new parameter, say δ, Marshall and Olkin (1997) introduced a new family of distributions namely Marshall - Olkin family of distributions with distribution function G(x) given by

$$G(x) = \frac{F(x)}{\delta + (1 - \delta)F(x)}, \quad x \in R \quad and \quad \delta > 0. \tag{6}$$

The corresponding survival function is

$$\bar{G}(x) = \frac{\delta \overline{F}(x)}{1 - (1 - \delta)\overline{F}(x)}, \quad x \in R \quad and \quad \delta > 0. \tag{7}$$

If $\delta = 1$, then G=F. If F has a density and hazard rate function, r_F, then by using the survival function $\overline{G}$, the density of G is given by

$$g(x;\delta) = \frac{\delta f(x)}{(1-(1-\delta)\overline{F}(x))^2}, \quad x \in R \quad and \quad \delta > 0 \tag{8}$$

and hazard rate function is

$$h(x;\delta) = \frac{r_F(x)}{1-(1-\delta)\overline{F}(x)}, \quad x \in R \quad and \quad \delta > 0.$$

Recently, many authors have developed various Marshall - Olkin distributions with respect to Gamma, Pareto, Weibull, Burr, Gumbel, Fr*é*chet, Rayleigh, Kumaraswamy, Linear Exponential, Lomax and other distributions. For details, see Jose and Alice (2003, 2004 a, 2004 b), Ghitany et.al (2005, 2007), Jayakumar and Mathew (2008), Jose et.al (2010, 2011), Jose and Rani (2013), Krishna et al. (2013 a, 2013 b), Jose and Remya (2015).

2 Generalized Lehmann Alternative Type II Family of Distributions

Let X be a random variable with cumulative distribution function (cdf) F(x). The survival function (sf) and probability density function (pdf) of X are denoted by $\bar{F}(x) = 1 - F(x)$ and f(x) respectively. By Lehmann Alternative Type II exponentiated family discussed in section (1.2), we can take the cdf as

$$T(x) = 1 - (\bar{F}(x))^{\beta}. \tag{9}$$

The sf is

$$\bar{T}(x) = (\bar{F}(x))^{\beta}. \tag{10}$$

The corresponding pdf is

$$t(x) = \beta f(x)(\bar{F}(x))^{\beta-1}. \tag{11}$$

Marshall and Olkin (1997) introduced a new method of adding a parameter to a family of distributions to develop the Marshall-Olkin family which is discussed in section (1.3) and cumulative distribution function is given in (6).

Applying (9) in (6), we get the new family of distributions called Generalized Lehmann Alternative Type II family with parameters (δ, β) with cdf

$$G(x) = \frac{T(x)}{1 - \bar{\delta}\overline{T}(x)}.$$

On simplification, we get

$$G(x) = \frac{1 - (\overline{F}(x))^{\beta}}{1 - \bar{\delta}(\overline{F}(x))^{\beta}}. \tag{12}$$

The sf is

$$\overline{G}(x) = 1 - \frac{1 - (\overline{F}(x))^{\beta}}{1 - \bar{\delta}(\overline{F}(x))^{\beta}}.$$

On simplification, we get

$$\overline{G}(x) = \frac{\delta(\bar{F}(x))^{\beta}}{[1 - \bar{\delta}(\bar{F}(x))^{\beta}]}. \tag{13}$$

The corresponding pdf is

$$g(x) = \frac{\delta\beta f(x)(\bar{F}(x))^{\beta-1}}{[1 - \bar{\delta}(\bar{F}(x))^{\beta}]^{2}}. \tag{14}$$

The new family is referred to as GLA2 (δ, β).

The hazard rate function is $h(x) = \frac{g(x)}{\bar{G}(x)}$ and is obtained as

$$h(x) = \frac{\beta f(x)}{\bar{F}(x)[1 - \bar{\delta}(\bar{F}(x))^{\beta}]}. \tag{15}$$

2.1 Maximum Likelihood Estimation

Let $X_1, X_2, ...X_n$ be a random sample of size n from GLA2 family, then the likelihood function is

$$L = (\delta\beta)^{n} \frac{\prod_{i=1}^{n} f(x_i \left[\overline{F}(x_i)\right]^{\beta-1})}{\prod_{i=1}^{n} \left[1 - \bar{\delta}\left[\overline{F}(x_i)\right]^{\beta}\right]^{2}}.$$

The log likelihood function is given by

$$logL = n\log(\delta\beta) + \sum_{i=1}^{n}\log f(x_i) + (\beta-1)\sum_{i=1}^{n}\log \overline{F}(x_i) - 2\sum_{i=1}^{n}\log\left[1-\bar{\delta}(\overline{F}(x_i))^{\beta}\right].$$

The partial derivatives of the log likelihood with respect to δ and β are obtained as

$$\frac{\partial \log L}{\partial \delta} = \frac{n}{\delta} - 2\sum_{i=1}^{n}\frac{(\overline{F}(x_i))^{\beta}}{\left[1-\bar{\delta}(\overline{F}(x_i))^{\beta}\right]}$$

and

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log \overline{F}(x_i) + 2\sum_{i=1}^{n}\frac{\bar{\delta}(\overline{F}(x_i))^{\beta}\log \overline{F}(x_i)}{\left[1-\bar{\delta}(\overline{F}(x_i))^{\beta}\right]}.$$

In order to estimate the parameters, we have to solve the normal equations

$$\frac{\partial \log L}{\partial \delta} = 0; \quad \frac{\partial \log L}{\partial \beta} = 0. \tag{16}$$

Since (16) cannot be solved analytically, numerical iteration technique is used to get a solution for the parameters δ and β. One may use the nlm package in R software to get the maximum likelihood estimator (MLE) for the parameters.

3 Some Special Generalized Lehmann Alternative Type II Models

In this section, we obtain some special GLA2 models using Uniform distribution and Kumaraswamy distribution. Also we derive their probability density function (pdf), cumulative density function (cdf) and quantile and the different shapes of density function and hazard rate function.

3.1 Generalized Lehmann Alternative Type II Uniform distribution

Let X follows Uniform distribution with parameter θ with pdf, cdf and survival function $g(x) = \frac{1}{\theta}$, $G(x) = \frac{x}{\theta}$ and $\bar{G}(x) = 1 - \frac{x}{\theta}$ respectively. If we apply the cdf,

survival function and pdf of Uniform distribution in the cdf, survival function and pdf of GLA2 family given in (12), (13) and (14), we get the cdf, survival function and pdf of the new distribution called Generalized Lehmann Alternative Type II Uniform distribution with parameters δ, β, θ and is denoted by GLA2U.
The cdf is

$$G(x) = \frac{1-(1-\frac{x}{\theta})^{\beta}}{1-\overline{\delta}(1-\frac{x}{\theta})^{\beta}} \qquad x, \delta, \beta, \theta > 0.$$

On simplification, we get

$$G(x) = \frac{\theta^{\beta}-(\theta-x)^{\beta}}{\theta^{\beta}-\overline{\delta}(\theta-x)^{\beta}} \qquad x, \delta, \beta, \theta > 0. \tag{17}$$

The corresponding sf $\overline{G}(x) = 1 - G(x)$ is

$$\overline{G}(x) = \frac{\delta(\theta-x)^{\beta}}{\theta^{\beta}-\overline{\delta}(\theta-x)^{\beta}} \qquad x, \delta, \beta, \theta > 0. \tag{18}$$

The corresponding pdf is

$$g(x) = \frac{\delta\beta\theta^{\beta}(\theta-x)^{\beta-1}}{[\theta^{\beta}-\overline{\delta}(\theta-x)^{\beta}]^2} \qquad x, \delta, \beta, \theta > 0. \tag{19}$$

The density plot for different values of the parameters are given in Fig 1

The hazard rate function is $h(x) = \frac{g(x)}{\overline{G}(x)}$ and is obtained as

$$h(x) = \frac{\beta\theta^{\beta}}{(\theta-x)[\theta^{\beta}-\overline{\delta}(\theta-x)^{\beta}]} \qquad x, \delta, \beta, \theta > 0.$$

The graph of h(x) is given in Fig 2. It gives J- shaped and bath tub shaped curves.

The u^{th} quantile of GLA2U distribution can be obtained by inverting $G(x) = u$ and is given by

$$x_u = \theta\left\{1-\left[\frac{(u-1)}{u\overline{\delta}+1}\right]^{\frac{1}{\beta}}\right\}, \tag{20}$$

where $0 < u < 1$.

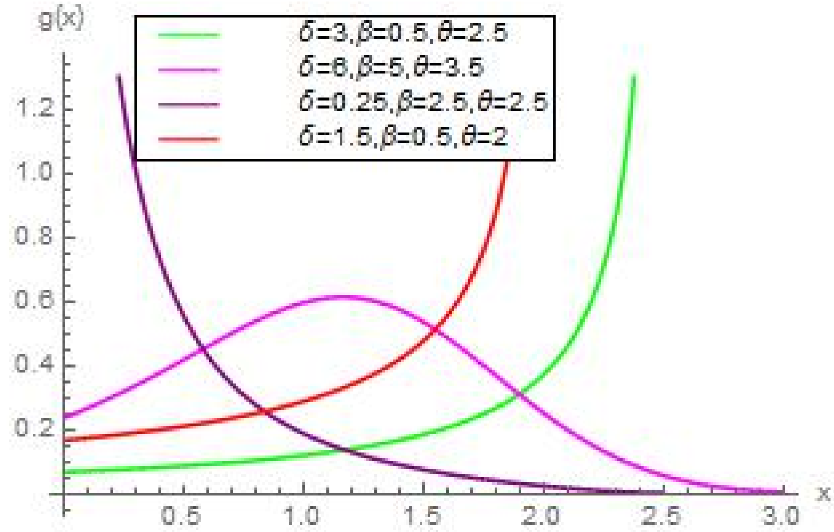

Figure 1: Probability density function of GLA2U for various values of δ, β and θ

3.2 Generalized Lehmann Alternative Type II Kumaraswamy Distribution

Let X follows Kumaraswamy distribution with parameters α and γ with cdf, survival function and pdf $F(x) = 1-(1-x^{\alpha})^{\gamma}$, $\bar{F}(x) = (1-x^{\alpha})^{\gamma}$ and $f(x) = \alpha\gamma x^{(\alpha-1)}(1-x^{\alpha})^{(\gamma-1)}$ respectively. If we apply the cdf, survival function and pdf of Kumaraswamy distribution in the cdf, survival function and pdf of Generalized Lehmann Alternative Type II family given in (12), (13) and (14), we get the cdf, survival function and pdf of the new distribution called Generalized Lehmann Alternative Type II Kumaraswamy distribution with parameters δ, β, α and γ is denoted by GLA2Kw.
The cdf is

$$G(x) = \frac{1-(1-x^{\alpha})^{\beta\gamma}}{1-\bar{\delta}(1-x^{\alpha})^{\beta\gamma}} \quad x, \delta, \beta, \alpha, \gamma > 0. \tag{21}$$

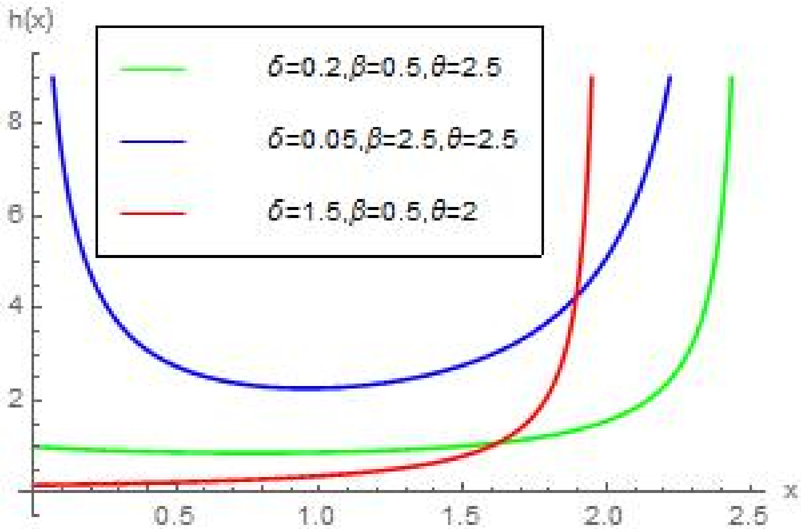

Figure 2: Hazard rate function of GLA2U for various values of δ, β and θ

The corresponding sf $\bar{G}(x) = 1 - G(x)$ is

$$\bar{G}(x) = \frac{\delta(1-x^{\alpha})^{\beta\gamma}}{1-\bar{\delta}(1-x^{\alpha})^{\beta\gamma}} \quad x, \delta, \beta, \alpha, \gamma > 0. \tag{22}$$

The corresponding pdf is

$$g(x) = \frac{\delta\beta\alpha\gamma x^{(\alpha-1)}(1-x^{\alpha})^{(\gamma-1)}(1-x^{\alpha})^{\gamma(\beta-1)}}{[1-\bar{\delta}(1-x^{\alpha})^{\beta\gamma}]^2} \quad x, \delta, \beta, \alpha, \gamma > 0. \tag{23}$$

The density plot for different values of the parameters are given in Figure 3.

The hazard rate function is $h(x) = \frac{g(x)}{\bar{G}(x)}$ and is obtained as

$$h(x) = \frac{\beta\alpha\gamma x^{(\alpha-1)}(1-x^{\alpha})^{(\gamma-1)}}{(1-x^{\alpha})^{\gamma}[1-\bar{\delta}(1-x^{\alpha})^{\beta\gamma}]} \quad x, \delta, \beta, \alpha, \gamma > 0. \tag{24}$$

The plot of h(x) for different values of the parameters are given in Figure 4. It shows increasing and bath tub shaped curves.

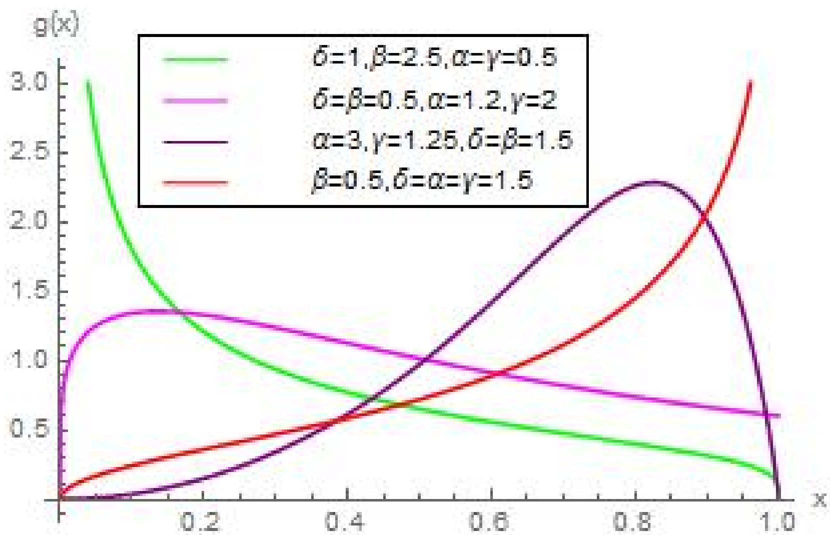

Figure 3: Probability density function of GLA2Kw for various values of δ, β, α and γ

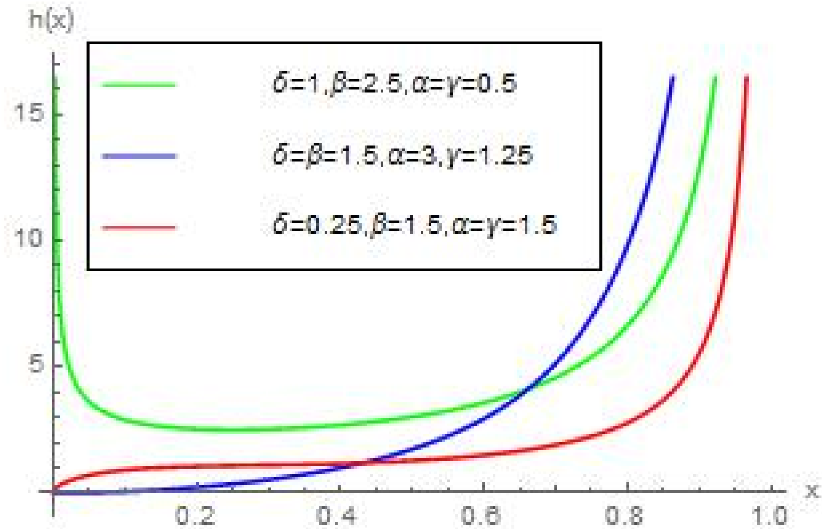

Figure 4: Hazard rate function of GLA2Kw for various values of δ, β, α and γ

The u^{th} quantile of GLA2Kw distribution can be obtained by inverting $G(x) = u$ and is given by

$$x_u = \left\{1 - \left[\frac{1-u}{1-u+u\delta}\right]^{\frac{1}{\beta\gamma}}\right\}^{\frac{1}{\alpha}} \tag{25}$$

where $0 < u < 1$.

3.3 Generalized Lehmann Alternative Type II Exponential Distribution

Exponential distribution plays a central role in analysis of lifetime or survival data, in part of their convenient statistical theory, their important lack of memory property and their constant hazard rates. In circumstances where the one-parameter family of exponential distributions is not sufficiently broad, a number of wider families such as the gamma, Weibull and Gompertz-Makeham distributions are in common use. Let $\overline{F}(x) = \mathrm{e}^{-\lambda x}, \quad x \geq 0$ is the survival function of exponential distribution, by

(9) we get the new distribution called Generalized Lehmann Alternative Type II Exponential (GLA2E) distribution with parameters (δ, β, λ) with cdf

$$G(x) = \frac{e^{\beta\lambda x} - 1}{e^{\beta\lambda x} - \bar{\delta}} \quad x \geq 0, \quad \lambda, \beta \ \ and \ \ \delta > 0, \bar{\delta} = 1 - \delta. \tag{26}$$

The survival function

$$\bar{G}(x) = \frac{\delta}{e^{\beta\lambda x} - \bar{\delta}} \quad x \geq 0, \quad \lambda, \beta \ \ and \ \ \delta > 0, \bar{\delta} = 1 - \delta. \tag{27}$$

Then the pdf is

$$g(x) = \frac{\delta\beta\lambda e^{\beta\lambda x}}{[e^{\beta\lambda x} - \bar{\delta}]^2} \quad x \geq 0, \quad \lambda, \ \ \beta \ \ and \ \ \delta > 0, \bar{\delta} = 1 - \delta. \tag{28}$$

The graph of g(x) is given in Fig 5.

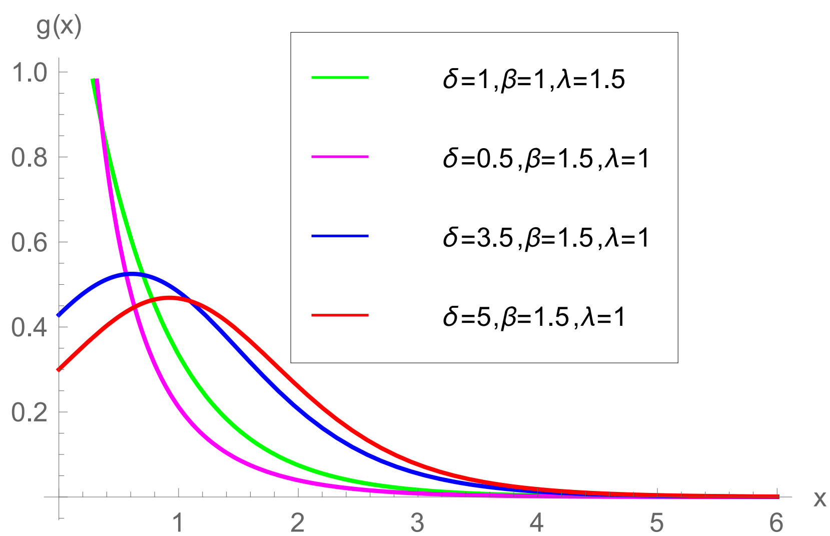

Figure 5: Probability density function of GLA2E (δ, β, λ) for various values of δ, β and λ

The hazard rate is

$$h(x) = \frac{\beta\lambda e^{\beta\lambda x}}{e^{\beta\lambda x} - \bar{\delta}} \quad x \geq 0, \quad \lambda, \ \ \beta \ \ and \ \ \delta > 0. \tag{29}$$

The graph of h(x) is given in Fig 6. It can be seen that the hazard rate is DFR for $\delta < 1$, and IFR for $\delta > 1$. Note that for $\delta = 1$, h(x)=1, showing constant failure rate. This establishes the wide applicability of the GLA2E distribution in reliability modeling.

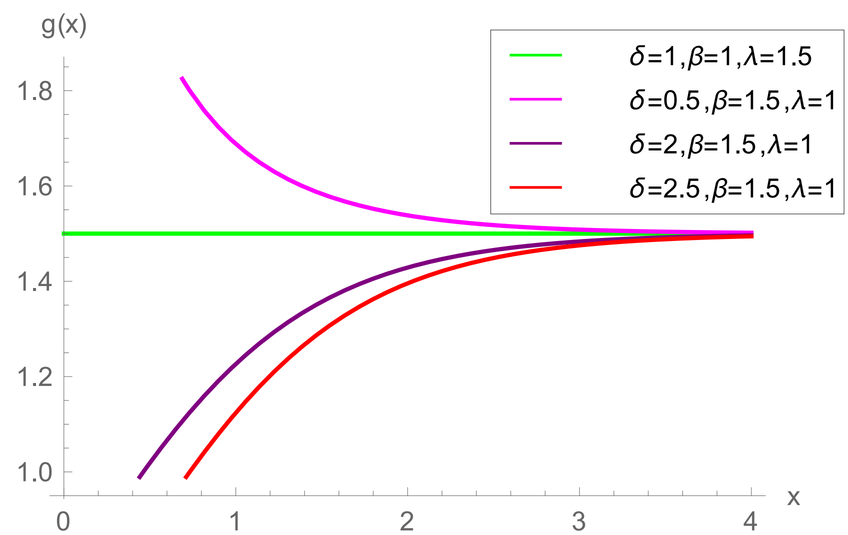

Figure 6: Hazard rate function of GLA2E (δ, β, λ) for various values of δ, β and λ

The u^{th} quantile is obtained by inverting the cdf given in (26).

$$x_u = \frac{1}{\beta\lambda} \log\left[\frac{u\overline{\delta} - 1}{u - 1}\right], \tag{30}$$

where U follows U(0,1).

3.3.1 Maximum Likelihood Estimation

Let $x_1, x_2, ..., x_n$ be a random sample of size n from $GLA2E(\delta, \beta, \lambda)$ distribution with pdf (28). The likelihood function is given by

$$L(\delta, \beta, \lambda) = \frac{(\delta\beta\lambda)^n e^{n\beta\lambda\bar{x}}}{\prod_{i=1}^{n}(e^{\beta\lambda x_i} - (1-\delta))^2}.$$

The log likelihood function is

$$logL = nlog(\delta\beta\lambda) + \sum_{i=1}^{n} \beta\lambda x_i - 2\sum_{i=1}^{n} log[e^{\beta\lambda x_i} - \bar{\delta}].$$

The MLE's of δ, β and λ are given by the solution of the three equations:

$$\frac{n}{\delta} - 2\sum_{i=1}^{n} \frac{1}{e^{\beta\lambda x_i} - \bar{\delta}} = 0, \qquad (31)$$

$$\frac{n}{\lambda} + n\beta\overline{x} - 2\sum_{i=1}^{n} \frac{\beta x_i e^{\beta\lambda x_i}}{e^{\beta\lambda x_i} - \bar{\delta}} = 0 \qquad (32)$$

and

$$\frac{n}{\beta} + n\lambda\overline{x} - 2\sum_{i=1}^{n} \frac{\lambda x_i e^{\beta\lambda x_i}}{e^{\beta\lambda x_i} - \bar{\delta}} = 0. \qquad (33)$$

When $\delta = 1$, the model reduces to exponential distribution. Then we get, $\hat{\lambda} = \frac{1}{\bar{x}}$

Here we show that the Generalized Lehmann Alternative Type II model of Exponential distribution can be a better model than the one parameter exponential model and Marshall- Olkin Extended Exponential model when it is fitted for the following data. The data represents the failure times of the air conditioning system of an airplane reported in Linhart and Zucchini (1986) and is given in Table 1.

Using R program we estimate the parameters and obtain log likelihood, K-S statistic and p-value. The results are given in Table 2. From the table we can observe that the p-value is greater for GLA2E distribution than that of Exponential distribution and Marshall-Olkin Extended Exponential distribution. So we can

Table 1: Failure times of the air conditioning system of an airplane

23	261	87	7	120	14	62	47	225	71
246	21	42	20	5	12	120	11	3	14
71	11	14	11	16	90	1	16	52	95

Table 2: Summary statistics for the failure time data of the air conditioning system of an airplane.

Model	Parameter	MLE	-log L	K-S statistic	p-value
Exponential	λ	0.0168	152.6297	0.213	0.132
MOEE	δ	0.4072	151.425	0.129	0.6978
	λ	0.0106			
GLA2E	δ	0.3803			
	λ	0.0039	151.42	0.123	0.7508
	β	2.6135			

conclude that GLA2E distribution is a better model than Exponential distribution and Marshall-Olkin Extended distribution for the failure time data. The P-P plot and Q-Q plot for the data is given in Figure 7.

4 Record Value Theory for Generalized Lehmann Alternative Type II Exponential Distribution

Let $X_1, X_2, ...$ be an infinite sequence of i.i.d. random variables having the same distribution as the (population) random variable X. An observation X_j will be called an upper record value (or simply a record), if its value exceeds that of all previous observations. Then X_j is a record if $X_j > X_i$ for every $i < j$. The time at which records appear are of interest. Let X_j be observed at time j. Then the record time sequence $\{T_n,\ n \geq 0\}$ is defined as $T_0 = 1$ with probability 1 and for $n \geq 1, T_n = \min\{j : X_j > X_{T_{n-1}}\}$.The record value sequence $\{R_n\}$ is then defined by $R_n = X_{T_n}; \quad n = 1, 2, ...$ Then R_n is called the n^{th} record. Let $g_{R_n}(x)$ denote

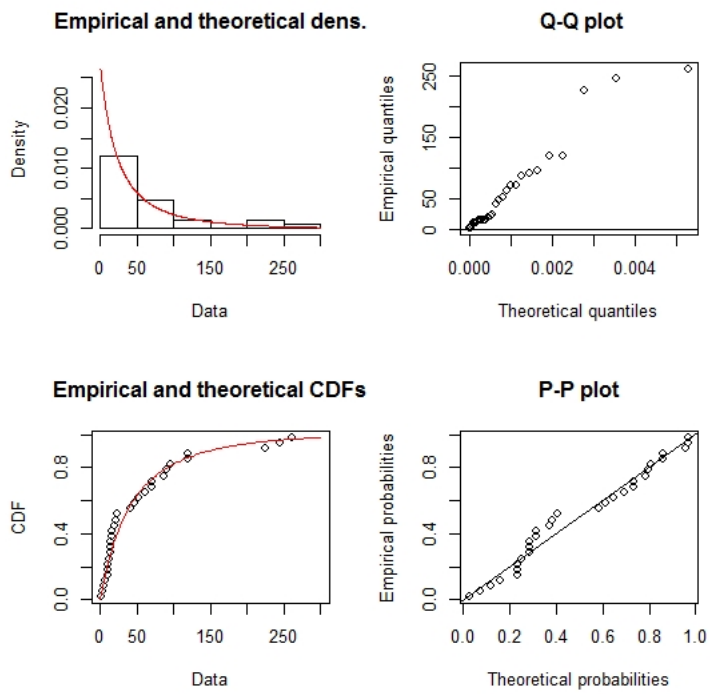

Figure 7: QQ plot and PP plot of GLA2E distribution

the p.d.f. of the n^{th} record. Then

$$g_{R_n}(x) = \frac{g(x)[-\log(\bar{G}(x))]^n}{n!}, -\infty < x < \infty. \tag{34}$$

The joint p.d.f. of a pair of records say R_m, R_n is given by

$$g_{R_m,R_n}(x,y) = \frac{[-\log \overline{G}(x)]^{m-1}}{(m-1)!} \frac{\left[-\log \frac{\overline{G}(y)}{\overline{G}(x)}\right]^{n-m-1}}{(n-m-1)!} \frac{g(x)g(y)}{1-G(x)}, -\infty < x < y < \infty. \tag{35}$$

Record data arise in a wide variety of practical situations such as industrial stress testing, meteorological analysis, hydrology, seismology, sporting and athletic events and oil mining surveys. In experiments related to these contexts measurements may be made sequentially and only the record values are observed. Usually the number of records of such experiments are considerably smaller than the complete sample size. This 'measurement saving' can be important when the measurements of these experiments are either costly or when the entire sample is destroyed.

Chandler (1952) introduced the study of record values and documented many of the basic properties of records. Arnold et al. (1998), Balakrishnan and Ahsanullah (1994), Balakrishnan et al. (1995), etc. have made significant contributions to the theory of records. Arnold et al. (1998) provide an excellent discussion on various results with respect to record values. Now we derive some record statistics with respect to Generalized Lehmann Alternative Type II Exponential distribution with $\lambda = 1$ for which the pdf is

$$g(x) = \frac{\delta\beta e^{\beta x}}{(e^{\beta x} - \bar{\delta})^2}, x > 0, \delta, \quad \beta > 0, \bar{\delta} = 1 - \bar{\delta} \tag{36}$$

By (34) the density function of the n^{th} record for $GLA2E(\delta, \beta, \lambda)$ distribution is given by

$$g_{R_n}(x) = \frac{\delta\beta \mathrm{e}^{\beta x}}{n![e^{\beta x} - \bar{\delta}]^2} \left[-\ln\left(\frac{\delta}{\mathrm{e}^{\beta x} - \bar{\delta}}\right)\right]^n, \qquad 0 < x < \infty \tag{37}$$

Then the single moment of n^{th} record statistic can be written as

$$\alpha_n = \frac{1}{\beta}\int_0^\infty ln(\bar{\delta} + \delta e^u)\frac{u^n}{n!}e^{-u}du. \tag{38}$$

Table 3: Mean of upper record values for $\beta = 1.5$

n	δ =1	δ =1.5	δ=2	δ =2.5	δ=3	δ =3.5	δ =4
1	1.3333	1.5434	1.7005	1.8265	1.9319	2.0225	2.1020
2	2	2.2410	2.4168	2.5554	2.6699	2.7675	2.8526
3	2.6667	2.9226	3.1038	3.2508	3.3691	3.4696	3.5568
4	3.3333	3.5965	3.7846	3.9311	4.0512	4.1528	4.2410
5	4	4.2668	4.4568	4.6044	4.7252	4.8275	4.9161
6	4.6667	4.9352	5.1261	5.2743	5.3955	5.4980	5.5869
7	5.3333	5.6028	5.7941	5.9426	6.0640	6.1666	6.2555

Theorem 4.1. *The single moment of n^{th} upper record value for $\delta > 0.5$ is given by*

$$\alpha_n = \frac{1}{\beta}\left\{\ln(\delta) + (n+1) - \sum_{i=1}^{\infty}\frac{k^i}{i(i+1)^{(n+1)}}\right\}, \quad where \quad k = 1 - \frac{1}{\delta}. \tag{39}$$

Proof From (38)and using the fact that $ln[1 - ke^{-u}] = -\sum_{i=1}^{\infty}\frac{k^i e^{-iu}}{i}$,

$$\alpha_n = \frac{1}{\beta}\left\{ln(\delta)\int_0^{\infty}\frac{u^n e^{-u}}{n!}du + \int_0^{\infty}\frac{u^{n+1}e^{-u}}{n!}du - \sum_{i=1}^{\infty}\frac{k^i}{i}\int_0^{\infty}\frac{e^{-(i+1)u}u^n}{n!}du\right\},$$

which on evaluation directly gives (39).

Using the result (39) the mean of record values from $GLA2E(\delta, \beta, \lambda)$ for different values of δ and for $\beta = 1.5$ and for $\delta = 1.5$ and for different values of β are evaluated and presented in Table 3 and Table 4.

Theorem 4.2. *The second single moment of n^{th} upper record value is*

$$\begin{aligned}\alpha_n^2 &= \frac{1}{\beta^2}\left\{\ln(\delta)^2 + (n+1)(n+2) + 2\ln(\delta) - 2(n+1)\sum_{i=1}^{\infty}\frac{k^i}{i(i+1)^{n+2}} - 2\ln(\delta)\right.\\ &\left.\times\sum_{i=1}^{\infty}\frac{k^i}{i(i+1)^{(n+1)}} + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{k^{i+j}}{ij(i+j+1)^{(n+1)}}\right\}.\end{aligned} \tag{40}$$

Proof: From (38) the 2^{nd} single moment of n^{th} record value is given by

Table 4: Mean of upper record values for $\delta = 1.5$

n	β =1	β =1.5	β=2	β =2.5	β=3	β =3.5	β =4
1	2.3150	1.5434	1.1575	0.9260	0.7717	0.6614	0.5788
2	3.3615	2.2410	1.6808	1.3446	1.1205	0.9604	0.8404
3	4.3839	2.9226	2.1919	1.7536	1.4613	1.2525	1.0960
4	5.3948	3.5965	2.6974	2.1579	1.7983	1.5414	1.3487
5	6.4002	4.2668	3.2001	2.5601	2.1334	1.8286	1.6000
6	7.4028	5.1261	3.7014	2.9611	2.4676	2.1151	1.8507
7	8.4042	5.6028	4.2021	3.3617	2.8014	2.4012	2.1010

$$\begin{aligned}
\alpha_n^2 &= \int_0^\infty \left\{\ln[\delta e^u(1-ke^{-u})]\right\}^2 \frac{u^n e^{-u}}{(n)!}du, \quad k = 1-\frac{1}{\delta} \\
&= (\ln\delta)^2 + (n+1)(n+2) + 2(n+1)\ ln\delta - 2(n+1)\sum_{i=1}^{\infty}\frac{k^i}{i(i+1)^{(n+2)}} - 2 \\
&\quad \times \ln\delta \sum_{i=1}^{\infty}\frac{k^i}{i(i+1)^{n+1}} + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{k^{i+j}}{ij}\int_0^\infty e^{-(i+j+1)u}\frac{u^n}{(n)!}du.
\end{aligned}$$

On simplification using the fact that $(a_1+a_2)^2 = \sum_{i=1}^{2}\sum_{j=1}^{2} a_i a_j$ we get (40).

By (35) the joint pdf of m^{th} and n^{th} record values of GLA2E (δ, β) distribution is given by

$$\begin{aligned}
g_{R_m,R_n}(x) &= \frac{\delta\beta^2\left[-\ln\left\{\dfrac{\delta}{\mathrm{e}^{\beta x}-(1-\delta)}\right\}\right]^m}{(m)!}\frac{1}{[\mathrm{e}^{\beta x}-(1-\delta)]}. \\
&\quad \times \frac{\left[-\ln\left\{\dfrac{\mathrm{e}^{\beta x}-(1-\delta)}{\mathrm{e}^{y}-(1-\delta)}\right\}\right]^{n-m-1}}{(n-m-1)!} \\
&\quad \times \frac{\mathrm{e}^{\beta y}}{[\mathrm{e}^{\beta y}-(1-\delta)]^2}, 0 < x < y < \infty.
\end{aligned}$$

Theorem 4.3. *For $1 \leq m \leq n$ the product moment*

$$\alpha_{m,n} = \frac{1}{\beta^2}\Bigg\{(\ln\delta)^2 + \ln\delta(m+n+2) + (m+1)(n+2) - [\ln\delta + (n-m)] \times \sum_{i=1}^{\infty}\frac{k^i}{i(i+1)^{m+1}} - (m+1)\sum_{i=1}^{\infty}\frac{k^i}{i(i+1)^{m+2}} - \ln\delta\sum_{i=1}^{\infty}\frac{k^j}{j(j+1)^{n+1}} - (m+1) \times \sum_{j=1}^{\infty}\frac{k^j}{j(j+1)^{n+2}} + \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{k^{(i+j)}}{ij(j+1)^{n-m}(i+j+1)^{(m+1)}}\Bigg\} \tag{41}$$

Proof:

$$\alpha_{m,n} = \frac{\delta\beta^2}{(m)!}\int_0^{\infty} x\left[-\ln\left(\frac{\delta}{e^{\beta x}-\bar{\delta}}\right)\right]^{m-1}\frac{e^{\beta x}}{e^{\beta x}-\bar{\delta}}\; I_x\; dx \tag{42}$$

where

$$I_x = \frac{1}{(n-m-1)!}\int_x^{\infty}\frac{ye^{\beta y}}{(e^{\beta y}-\bar{\delta})^2}\left[-\ln\left(\frac{e^{\beta x}-\bar{\delta}}{e^{\beta y}-\bar{\delta}}\right)\right]^{(n-m-1)} dy$$

now making use of the transformation $u = -\ln\left(\frac{e^{\beta x}-\bar{\delta}}{e^{\beta y}-\bar{\delta}}\right)$

and writing $\ln\left[1-\left(\frac{\delta-1}{e^x-\bar{\delta}}\right)e^{-u}\right] = -\sum_{i=1}^{\infty}\left(\frac{\delta-1}{e^x-\bar{\delta}}\right)^i\frac{e^{-iu}}{i}$ we get

$$I_x = \frac{1}{\beta^2(e^{\beta x}-\bar{\delta})}\left[\ln(e^{\beta x}-\bar{\delta}) + (n-m) - \sum_{i=1}^{\infty}\left(\frac{\delta-1}{e^x-\bar{\delta}}\right)^i\frac{1}{i(i+1)^{n-m}}\right]$$

substituting the expression of I_x in (42) and using the transformation $t = -\ln\left(\frac{\delta}{e^{\beta x}-\bar{\delta}}\right)$ yields (41). Using (39), (40) and (41) numerical values of variance and covariance of upper record values are obtained by MATLAB program for $\beta = 1.5$ and various values of δ and is presented in Table 5.

4.1 Estimation of the location and scale parameters

In industry experiments, the number of measurements can be made lesser if the record values are observed instead of complete sample for estimation of parameters. There are also situations in which an observation is stored if it is a record value. This includes studies in meteorology, hydrology, seismology athletic events and mining.

Table 5: Variance and covariance of upper record values for $\beta = 1.5$

m	n	$\delta = 1$	$\delta = 1.5$	$\delta = 2$	$\delta = 2.5$	$\delta = 3$	$\delta = 3.5$	$\delta = 4$
1	1	0.8889	0.9747	1.0295	1.0682	1.0973	1.1200	1.1384
	2	0.8889	0.9516	0.9900	1.0162	1.0354	1.0502	1.0619
	3	0.8889	0.9406	0.9718	0.9929	1.0083	1.0200	1.0293
	4	0.8889	0.9353	0.9632	0.9820	0.9957	1.0062	1.0144
	5	0.8889	0.9327	0.9591	0.9768	0.9898	0.9996	1.0074
	6	0.8889	0.9315	0.9570	0.9743	0.9869	0.9965	1.0041
	7	0.8889	0.9308	0.9560	0.9731	0.9855	0.9949	1.0024
2	2	1.3333	1.3943	1.4298	1.4532	1.4699	1.4825	1.4923
	3	1.3333	1.3783	1.4039	1.4204	1.4321	1.4408	1.4475
	4	1.3333	1.3706	1.3916	1.4051	1.4146	1.4217	1.4271
	5	1.3333	1.3669	1.3857	1.3978	1.4063	1.4126	1.4175
	6	1.3333	1.3650	1.3828	1.3943	1.4023	1.4083	1.4129
	7	1.3333	1.3641	1.3814	1.3925	1.4003	1.4061	1.4106
3	3	1.7778	1.8171	1.8387	1.8524	1.8619	1.8689	1.8743
	4	1.7778	1.8070	1.8228	1.8327	1.8396	1.8446	1.8484
	5	1.7778	1.8020	1.8151	1.8233	1.8289	1.8330	1.8362
	6	1.7778	1.7996	1.8113	1.8187	1.8237	1.8274	1.8303
	7	1.7778	1.7984	1.8095	1.8164	1.8212	1.8247	1.8274
4	4	2.2222	2.2463	2.2590	2.2670	2.2724	2.2763	2.2793
	5	2.2222	2.2401	2.2496	2.2554	2.2594	2.2622	2.2644
	6	2.2222	2.2371	2.2450	2.2498	2.2531	2.2554	2.2572
	7	2.2222	2.2356	2.2427	2.2470	2.2500	2.2521	2.2537
5	5	2.6667	2.6809	2.6883	2.6928	2.6959	2.6981	2.6998
	6	2.6667	2.6773	2.6828	2.6862	2.6884	2.6901	2.6913
	7	2.6667	2.6755	2.6801	2.6829	2.6848	2.6861	2.6871
6	6	3.1111	3.1193	3.1236	3.1261	3.1279	3.1291	3.1300
	7	3.1111	3.1173	3.1204	3.1223	3.1236	3.1245	3.1252
7	7	3.5556	3.5602	3.5626	3.5640	3.5650	3.5657	3.5662

Recently much studies have been made on parametric and non parametric inferences based on record values. Raquab (2002) obtained inference for generalised exponential distribution based on record statistics. Soliman et al. made a comparison of Bayesian and non-Bayesian estimates using record statistics from Weibull model. Sultan et al. (2008) obtained the estimation from record values and predicted future records for gamma distribution. Sultan (2010) discussed different methods of estimation based on record values from inverse Weibull distribution.
Consider the general location-scale family of distributions with cdf $F(x,\mu,\sigma) = F(\frac{x-\mu}{\sigma})$ and pdf $f(x,\mu,\sigma) = \frac{1}{\sigma}f(\frac{x-\mu}{\sigma})$ and assume that the upper record values $R_1, R_2, ...R_n$ are available. Then **BLUE**'s of μ and σ are given respectively by, (see Balakrishnan and Cochen, 1991)

$$\mu^* = \frac{\alpha^T\boldsymbol{\Sigma}^{-1}\alpha\mathbf{1}^T\boldsymbol{\Sigma}^{-1} - \alpha^T\Sigma^{-1}\mathbf{1}\alpha^T\boldsymbol{\Sigma}^{-1}}{(\alpha^T\boldsymbol{\Sigma}^{-1}\alpha)(\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\alpha^T\boldsymbol{\Sigma}^{-1}\mathbf{1})^2}\mathbf{R} = \sum_{i=1}^{n} a_i R_i \tag{43}$$

$$\sigma^* = \frac{\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}\alpha^T\boldsymbol{\Sigma}^{-1} - \mathbf{1}^T\boldsymbol{\Sigma}^{-1}\alpha\mathbf{1}^T\boldsymbol{\Sigma}^{-1}}{(\alpha^T\boldsymbol{\Sigma}^{-1}\alpha)(\mathbf{1}^T\boldsymbol{\Sigma}^{-1}\mathbf{1}) - (\alpha^T\boldsymbol{\Sigma}^{-1}\mathbf{1})^2}\mathbf{R} = \sum_{i=1}^{n} b_i R_i \tag{44}$$

where α denotes the column vector of the expected values of observed upper record values from the distribution F(x), $\boldsymbol{\Sigma}$ denotes the variance-covariance matrix of the record values from the distribution F(x), and $\mathbf{1}$ is a column vector of dimension n with all its entries as 1.
The three parameter Generalized Lehmann Alternative Type II exponential distribution has the probability density function given by

$$g(y) = \frac{\delta e^{\frac{(y-\mu)}{\sigma}}}{\sigma(e^{(\frac{(y-\mu)}{\sigma})} - \bar{\delta})^2}, y > \mu, \delta, \sigma > 0,$$

where δ, μ and σ are the shape,location and scale parameters respectively. By making use of means, variances and covariances presented in Table 3, Table 4, and Table 5, we calculate the coefficients of **BLUE**s a_i and b_i, i=1,2,...n for different values of shape parameter δ and n and presented in Table 6 and Table 7. It can be noted from these tables that $\sum_{i=1}^{n} a_i = 1$ and $\sum_{i=1}^{n} b_i = 0$

Table 6: Coefficients of the BLUE of μ for $\beta = 1.5$

n	$\delta = 1$	$\delta = 1.5$	$\delta = 2$	$\delta = 2.5$	$\delta = 3$	$\delta = 3.5$	$\delta = 4$
2	2.9999 -1.9999	3.2124 -2.2124	3.3740 -2.3740	3.5058 -2.5058	3.6178 -2.6178	3.7148 -2.7148	3.8004 -2.8004
3	2.0000 -0.0002 -0.9998	2.1061 0.0262 -1.1323	2.1921 0.0403 -1.2324	2.2525 0.0612 -1.3137	2.3077 0.0750 -1.3827	2.3562 0.0854 -1.4416	2.3990 0.0948 -1.4938
4	1.6667 -0.0001 0.0000 -0.6666	1.7318 0.0259 0.0087 -0.7664	1.7818 0.0410 0.0220 -0.8448	1.8228 0.0597 0.0187 -0.9012	1.8574 0.0722 0.0221 -0.9517	1.8880 0.0819 0.0252 -0.9951	1.9151 0.0908 0.0271 -1.0330
5	1.5000 -0.0001 0.0000 0.0000 -0.4999	1.5419 0.0260 0.0096 0.0030 -0.5805	1.5750 0.0414 0.0217 0.0026 -0.6407	1.6024 0.0587 0.0209 0.0066 -0.6886	1.6259 0.0707 0.0243 0.0079 -0.7288	1.6469 0.0800 0.0277 0.0080 -0.7626	1.6655 0.0884 0.0300 0.0089 -0.7928
6	1.4000 -0.0001 0.0000 0.0000 0.0001 -0.4000	1.4268 0.0259 0.0102 0.0036 0.0008 -0.4673	1.4488 0.0417 0.0215 0.0042 0.0013 -0.5175	1.4674 0.0582 0.0220 0.0079 0.0025 -0.5580	1.4837 0.0699 0.0255 0.0094 0.0023 -0.5908	1.4986 0.0788 0.0291 0.0097 0.0033 -0.6195	1.5118 0.0869 0.0316 0.0107 0.0034 -0.6444
7	1.3333 -0.0001 0.0000 0.0000 0.0001 -0.0001 -0.3332	1.3493 0.0257 0.0106 0.0041 0.0011 0.0014 -0.3922	1.3635 0.0418 0.0215 0.0051 0.0019 0.0009 -0.4347	1.3760 0.0576 0.0228 0.0087 0.0031 0.0009 -0.4691	1.3872 0.0691 0.0266 0.0103 0.0033 0.0010 -0.4975	1.3978 0.0779 0.0302 0.0109 0.0040 0.0006 -0.5214	1.4074 0.0858 0.0326 0.0121 0.0041 0.0005 -0.5425

Table 7: Coefficients for the BLUE of σ for $\beta = 1.5$

n	$\delta=1$	$\delta=1.5$	$\delta=2$	$\delta=2.5$	$\delta=3$	$\delta=3.5$	$\delta=4$
2	-1.4999 1.4999	-1.3671 1.3671	-1.2942 1.2942	-1.2466 1.2466	-1.2126 1.2126	-1.1877 1.1877	-1.1678 1.1678
3	-1.4999 0.0003 0.4996	-1.3824 -0.0013 1.3837	-1.3162 0.0019 1.3143	-1.2729 -0.0058 1.2787	-1.2415 -0.0085 1.2500	-1.2181 -0.0103 1.2284	-1.1996 -0.0117 1.2113
4	-1.4999 0.0003 -0.0001 1.4997	-1.3897 -0.0014 0.0069 1.3842	-1.3269 0.0018 -0.0015 1.3266	-1.2850 -0.0064 0.0114 1.2800	-1.2546 -0.0090 0.0119 1.2517	-1.2319 -0.0109 0.0122 1.2306	-1.2137 -0.0127 0.0125 1.2139
5	-1.4999 0.0003 -0.0001 0.0001 1.4996	-1.3930 -0.0021 0.0072 0.0073 1.3806	-1.3321 0.0016 -0.0013 0.0157 1.3161	-1.2907 -0.0067 0.0111 0.0119 1.2744	-1.2609 -0.0094 0.0120 0.0128 1.2455	-1.2384 -0.0113 0.0121 0.0138 1.2238	-1.2203 -0.0132 0.0122 0.0140 1.2073
6	-1.4999 0.0003 -0.0001 0.0001 -0.0003 1.4999	-1.3946 -0.0022 0.0069 0.0074 0.0058 1.3767	-1.3343 0.0012 -0.0011 0.0154 0.0087 1.3101	-1.2935 -0.0071 0.0113 0.0119 0.0088 1.2686	-1.2638 -0.0098 0.0123 0.0125 0.0102 1.2386	-1.2415 -0.0118 0.0123 0.0138 0.0099 1.2173	-1.2236 -0.0134 0.0121 0.0141 0.0101 1.2007
7	-1.4999 0.0003 -0.0001 0.0001 -0.0003 0.0005 1.4994	-1.3957 -0.0019 0.0068 0.0074 0.0061 0.0023 1.3750	-1.3355 0.0012 -0.0014 0.0157 0.0084 0.0050 1.3066	-1.2951 -0.0069 0.0113 0.0119 0.0088 0.0060 1.2640	-1.2654 -0.0096 0.0119 0.0128 0.0097 0.0065 1.2341	-1.2430 -0.0116 0.0121 0.0135 0.0101 0.0071 1.2118	-1.2251 -0.0135 0.0121 0.0138 0.0104 0.0075 1.1948

The variances and covariance of the BLUE's of μ and σ are given by (see Balakrishnan and Cochen,1991)

$$Var(\mu^*) = \sigma^2 \left\{ \frac{\alpha^T \mathbf{\Sigma}^{-1} \alpha}{(\alpha^T \mathbf{\Sigma}^{-1} \alpha)(\mathbf{1}^T \mathbf{\Sigma}^{-1} \mathbf{1}) - (\alpha^T \mathbf{\Sigma}^{-1} \mathbf{1})^2} \right\} = \sigma^2 V_1$$

$$Var(\sigma^*) = \sigma^2 \left\{ \frac{\mathbf{1}^T \mathbf{\Sigma}^{-1} \mathbf{1}}{(\alpha^T \mathbf{\Sigma}^{-1} \alpha)(\mathbf{1}^T \mathbf{\Sigma}^{-1} \mathbf{1}) - (\alpha^T \mathbf{\Sigma}^{-1} \mathbf{1})^2} \right\} = \sigma^2 V_2$$

$$Cov(\mu^*, \sigma^*) = \sigma^2 \left\{ \frac{-\alpha^T \mathbf{\Sigma}^{-1} \mathbf{1}}{(\alpha^T \mathbf{\Sigma}^{-1} \alpha)(\mathbf{1}^T \mathbf{\Sigma}^{-1} \mathbf{1}) - (\alpha^T \mathbf{\Sigma}^{-1} \mathbf{1})^2} \right\} = \sigma^2 V_3$$

Using these results Variance and covariances of the BLUE's of μ and σ can be obtained in terms of σ^2 and is presented in Table 8.

Example: Consider a simulated data of failure times which follow GLA2E distribution with $\alpha = \delta = 1.5 \; and \; \lambda = 1$,
1.3517, 1.8239, 0.1316, 1.18816, 0.8503, 0.1002, 0.3045, 0.6889, 2.3664, 2.4953, 0.1649, 2.6148, 2.3610, 0.5877, 1.2983, 0.1477, 0.4927, 1.9005, 1.2699, 2.3988, 0.9000, 0.0360, 1.4968, 2.0675, 0.9518, 1.1593, 1.1168, 0.4514, 0.8994, 0.1799, 1.0178, 0.0321, 0.3026, 0.0467, 0.997, 1.3860, 0.9900, 0.3524, 2.2593, 0.0348, 0.5173, 0.4368, 1.1830, 1.2803, 0.1975.
The observed upper record values are then,
1.3517, 1.8239, 1.8816, 2.3664, 2.4953, 2.6148.
With n=6, $\alpha = \delta = 1.5 \; and \; \lambda = 1$, the BLUE's of μ andσ can be computed using (43), (44) and Tables 6 and 7. The estimates are $\mu^* = 0.7837$ and $\sigma^* = 1.7557$
The corresponding variances and covariance of μ^* and σ^* can be obtained from Table 8
$Var(\mu^*) = 1.4757, Var(\sigma^*) = 0.9132$ and $Cov(\mu^* \sigma^*) = -0.3122$.

5 Confidence interval

Through the pivotal quantities

$$R_1 = \frac{\mu^* - \mu}{\sigma \sqrt{V_1}}, \quad R_2 = \frac{\sigma^* - \sigma}{\sigma \sqrt{V_2}} \quad \text{and} \quad R_3 = \frac{\mu^* - \mu}{\sigma^* \sqrt{V_1}}$$

Table 8: Variance and covariances of the BLUE's of μ and σ in terms of σ^2 for $\beta = 1.5$

n	$\delta = 1$	$\delta = 1.5$	$\delta = 2$	$\delta = 2.5$	$\delta = 3$	$\delta = 3.5$	$\delta = 4$
2	2.6662 0.9998 -1.3330	3.3570 0.9129 -1.5104	3.9183 0.8660 -1.6437	4.3993 0.8363 -1.7524	4.8230 0.8157 -1.8447	5.1994 0.8004 -1.5834	5.5422 0.7887 -1.9931
3	1.7777 0.9998 -0.6666	2.1912 0.9131 -0.7636	2.5267 0.8666 -0.8395	2.7977 0.8373 -0.8945	3.0384 0.8170 -0.9435	3.2512 0.8020 -0.9854	3.4433 0.7904 -1.0224
4	1.4814 0.9998 -0.4444	1.7968 0.9132 -0.5137	2.0436 0.8668 -0.5651	2.2490 0.8375 -0.6064	2.4255 0.8173 -0.6408	2.5804 0.8023 -0.6701	2.7196 0.7908 -0.6960
5	1.3333 0.9998 -0.3333	1.5968 0.9132 -0.3880	1.8000 0.8668 -0.4284	1.9676 0.8376 -0.4608	2.1103 0.8173 -0.4876	2.2351 0.8024 -0.5106	2.3464 0.7909 -0.5306
6	1.2444 0.9998 -0.2666	1.4757 0.9132 -0.3122	1.6517 0.8668 -0.3457	1.7952 0.8376 -0.3724	1.9170 0.8173 -0.3946	2.0228 0.8024 -0.4134	2.1167 0.7910 -0.4299
7	1.1852 0.9998 -0.2222	1.3939 0.9132 -0.2612	1.5512 0.8668 -0.2900	1.6784 0.8376 -0.3128	1.7857 0.8173 -0.3318	1.8786 0.8025 -0.3479	1.9607 0.7910 -0.3619

where μ^* and σ^* are the BLUE's of μ and σ we construct confidence interval for the location and scale parameters . We use R_1 and R_3 to construct CIs for μ when σ is known and when σ is unknown respectively while R_3 is used to construct CI's for σ. The construction of CI's require the percentage points of R_1, R_2 and R_3 which is obtained by using the BLUE's μ^* and σ^* via Monte carlo simulation based on 10000 runs and are presented in Table 9, 10 and 11 respectively.

6 Application

Now we apply the inference procedure discussed in the previous section to upper records of simulated data sets of size n=2,3,4,5,6 and 7 (with $\mu = 0, \sigma = 1$ and $\beta = 1.5$ *and* $\delta = 2$). The BLUE's are calculated using Tables 6 and 7 and is presented in Table 12

Using the BLUE's given in Table 12 and the percentage points of R_1 and R_3 we construct 95% confidence interval for μ when σ known and σ unknown respectively through the formulae,

$$P(\mu^* - \sigma\sqrt{V_1}R_1(97.5) \leq \mu \leq \mu^* - \sigma\sqrt{V_1}R_1(2.5)) = 95\%$$

$$P(\mu^* - \sigma^*\sqrt{V_1}R_3(97.5) \leq \mu \leq \mu^* - \sigma^*\sqrt{V_1}R_3(2.5)) = 95\%$$

We also construct confidence interval for σ using percentage points of R_2 through the formula

$$P(\frac{\sigma^*}{1 + \sqrt{V_2}R_2(97.5)} \leq \sigma \leq \frac{\sigma^*}{1 + \sqrt{V_2}R_2(2.5)}) = 95\%$$

The result is presented in table 13.

7 Prediction for Future Records

Prediction of future records becomes a problem of great interest. For example,while studying the record rainfall or snowfall,having observed the record values until the

Table 9: Simulated percentage points of R_1

α	n	0.5%	2.5%	5%	95%	97.5%	99.5%
1	2	-5.6048	-4.3647	-3.6886	1.0680	1.2553	1.6490
	3	-3.7185	-2.8144	-2.4643	0.6801	0.9140	1.3233
	4	-2.7703	-2.1807	-1.8877	0.4046	0.6224	0.9843
	5	-2.1528	-1.6737	-1.4879	0.2438	0.4602	0.7969
	6	-1.8005	-1.4185	-1.2617	0.1494	0.3165	0.6166
	7	-1.5893	-1.2325	-1.0969	0.0862	0.2277	0.5063
1.5	2	-5.6466	-4.3540	-3.7946	1.0855	1.2716	1.6391
	3	-3.7999	-2.9755	-2.6006	0.7355	0.9606	1.3155
	4	-2.8989	-2.2755	-2.0017	0.4559	0.6564	1.0353
	5	-2.3951	-1.8908	-1.6573	0.2899	0.4698	0.8471
	6	-1.9860	-1.5645	-1.3872	0.1933	0.3909	0.7205
	7	-1.7495	-1.3776	-1.2197	0.1200	0.2753	0.5728
2	2	-5.6107	-4.3902	-3.8306	1.0717	1.2635	1.5995
	3	-3.9064	-3.0607	-2.6715	0.7296	0.9569	1.3151
	4	-3.1229	-2.4339	-2.1276	0.5083	0.7113	1.1249
	5	-2.4654	-1.9295	-1.7106	0.3350	0.5625	0.9132
	6	-2.1414	-1.6892	-1.4868	0.2611	0.4199	0.7161
	7	-1.8242	-1.4775	-1.3091	0.1706	0.3485	0.6791
2.5	2	-5.8927	-4.4976	-3.8838	1.0709	1.2466	1.5387
	3	-4.0531	-3.0908	-2.7025	0.7249	0.9657	1.2860
	4	- 3.0235	-2.4615	-2.1567	0.5309	0.7412	1.1419
	5	-2.6150	-2.0163	-1.7838	0.3883	0.5817	0.9316
	6	-2.2246	-1.7543	-1.5669	0.2762	0.4315	0.7423
	7	-1.9673	-1.5583	-1.3835	0.1807	0.3479	0.6522
3	2	-5.8003	-4.4627	-3.9436	1.0356	1.2033	1.5224
	3	-4.1028	-3.2689	-2.8776	0.7613	0.9902	1.3394
	4	-3.1483	-2.5294	-2.2474	0.5344	0.7431	1.1134
	5	-2.6157	-2.0956	-1.8573	0.3815	0.5995	0.9595
	6	-2.3164	-1.8209	-1.5990	0.3059	0.4818	0.7932
	7	-1.9585	-1.5881	-1.4197	0.2162	0.3862	0.7111
3.5	2	-6.2370	-4.6718	-4.0112	1.0674	1.2328	1.5462
	3	-4.1166	-3.1950	-2.8594	0.7699	1.0098	1.3219
	4	-3.2041	-2.5486	-2.2757	0.5444	0.7572	1.1057
	5	-2.7375	-2.1398	-1.9060	0.3761	0.6090	0.9719
	6	-2.4066	-1.8791	-1.6463	0.3256	0.5062	0.8187
	7	-2.0798	-1.6544	-1.4697	0.2469	0.4224	0.7090
4	2	-6.0906	-4.6749	-4.0870	1.0676	1.2242	1.4689
	3	-4.2192	-3.2277	-2.8414	0.7766	0.9953	1.3071
	4	-3.2869	-2.6134	-2.3116	0.5683	0.7820	1.1489
	5	-2.7192	-2.1562	-1.9143	0.4332	0.6245	0.9703
	6	-2.3595	-1.9166	-1.7047	0.3404	0.5247	0.8608
	7	-2.1023	-1.6707	-1.4857	0.2233	0.4076	0.7151

Table 10: Simulated percentage points of R_2

α	n	0.5%	2.5%	5%	95%	97.5%	99.5%
1	2	-0.9838	-0.8978	-0.8094	3.9539	4.6854	6.2236
	3	-2.9830	-2.5203	-2.2619	0.5187	0.7560	1.3453
	4	-0.3644	0.0518	0.3188	4.7602	5.5493	7.2161
	5	-0.1531	0.3424	0.5932	4.8116	5.4976	7.0743
	6	0.1207	0.5766	0.8148	4.9689	5.5365	7.2471
	7	0.2645	0.6930	0.9076	5.0563	5.7481	7.4083
1.5	2	-1.0320	-0.9571	-0.8635	3.8172	4.5124	6.0503
	3	-0.7996	-0.4741	-0.2231	4.4816	5.1672	6.6805
	4	-0.4557	0.0145	0.2936	4.7694	5.4141	6.9187
	5	-0.1041	0.3609	0.6511	4.9347	5.6011	7.2818
	6	0.0580	0.5663	0.8464	4.9561	5.6505	7.0729
	7	0.3164	0.7543	1.0037	5. 0623	5.7650	7.2539
2	2	-1.0522	-0.9774	-0.8833	3.7763	4.4201	5.9488
	3	-0.8191	-0.4983	-0.2563	4.4266	5.1720	6.7478
	4	-0.4811	-0.0344	0.2837	4.7767	5.4815	7.1890
	5	-0.1934	0.3373	0.6073	4.8510	5.4855	6.9931
	6	0.1243	0.5631	0.8653	5.0114	5.6598	7.2408
	7	0.3080	0.7478	1.0209	5.0818	5.7963	7.2729
2.5	2	-1.0761	-1.0034	-0.9050	3.7554	4.4691	5.9813
	3	-0.8322	-0.4708	-0.2281	4.4342	5.0509	6.7658
	4	-0.4872	-0.0288	0.2877	4.7682	5.4150	6.7047
	5	-0.1494	0.3359	0.6389	4.8793	0.5533	7.1673
	6	0.1229	0.5822	0.8625	5.0375	5.6685	7.1661
	7	0.3350	0.8027	1.0718	5.1550	5.7967	7.2462
3	2	-1.0899	-1.0018	-0.9140	3.7177	4.3137	5.6924
	3	-0.8224	-0.5342	-0.2882	4.5343	5.2340	6.6902
	4	-0.4637	-0.0462	0.2718	4.8124	5.4531	6.8974
	5	-0.1437	0.3301	0.6146	4.9329	5.5515	6.8813
	6	0.1154	0.5954	0.8826	5.0871	5.7683	7.1481
	7	0.3127	0.8421	1.1048	5.1286	5.7708	7.2325
3.5	2	-1.1011	-1.0315	-0.9306	3.7736	4.4235	6.1671
	3	-0.8570	-0.5157	-0.2688	4.4775	5.0794	6.6519
	4	-0.4731	-0.0485	0.2204	4.7681	5.3593	6.8074
	5	-0.1657	0.3277	0.6398	4.9728	5.6169	7.1989
	6	0.1388	0.5945	0.9078	5.0823	5.7973	7.4037
	7	0.2979	0.8223	1.0760	5.2012	5.8126	7.2913
4	2	-1.1098	-1.0405	-0.9559	3.7920	4.4534	5.8867
	3	-0.8337	-0.5448	-0.3060	4.3638	5.0319	6.5454
	4	-0.5067	-0.0460	0.2424	4.7966	5.4323	6.8444
	5	-0.1765	0.3169	0.6245	4.9266	5.5686	6.9046
	6	0.0889	0.5909	0.8859	5.2025	5.8539	7.2088
	7	0.3425	0.8379	1.1365	5.1715	5.8387	7.3866

Table 11: Simulated percentage points of R_3

δ	n	0.5%	2.5%	5%	95%	97.5%	99.5%
1	2	-0.8146	-0.8057	-0.7950	5.2623	11.2520	77.3761
	3	-37.8845	-9.6445	-5.5812	3.5601	7.8030	47.7056
	4	-0.3645	-0.3616	-0.3579	0.2737	0.5256	1.3984
	5	-0.2881	-0.2860	-0.2830	0.1349	0.2979	0.7678
	6	-0.2387	-0.2369	-0.2346	0.0748	0.1703	0.4544
	7	-0.2038	-0.2024	-0.2007	0.0370	0.1095	0.3074
1.5	2	-0.8807	-0.8699	-0.8565	6.1249	13.1325	83.0720
	3	-0.5461	-0.5394	-0.5315	0.9181	1.5644	4.7084
	4	-0.4058	-0.3999	-0.3950	0.3298	0.6046	1.5776
	5	-0.3256	-0.3208	-0.3167	0.1638	0.3204	0.8235
	6	-0.2733	-0.2694	-0.2660	0.0971	0.2207	0.5773
	7	-0.2361	-0.2329	-0.2300	0.0541	0.1424	0.3666
2	2	-0.9223	-0.9085	-0.8927	5.9957	13.1293	60.6882
	3	-0.5813	-0.5706	-0.5608	0.9346	1.6818	4.6838
	4	-0.4348	-0.4260	-0.4183	0.3842	0.6907	1.8459
	5	-0.3518	-0.3445	-0.3390	0.1945	0.3876	0.9591
	6	-0.2979	-0.2919	-0.2872	0.1323	0.2514	0.5490
	7	-0.2581	-0.2535	-0.2494	0.0802	0.1820	0.4667
2.5	2	-0.9554	-0.9397	-0.9209	6.1792	13.6103	75.7308
	3	-0.6022	-0.5899	-0.5798	0.8775	1.6191	4.5777
	4	-0.4548	-0.4452	-0.4372	0.3880	0.7019	1.8229
	5	-0.3713	-0.3627	-0.3562	0.2331	0.4065	0.9663
	6	-0.3147	-0.3076	-0.3025	0.1400	0.2609	0.5943
	7	-0.2746	-0.2689	-0.2637	0.0834	0.1789	0.4137
3	2	-0.9783	-0.9606	-0.9432	5.8539	12.0516	76.3009
	3	-0.6211	-0.6079	-0.5967	1.0201	1.8443	4.7492
	4	-0.4718	-0.4613	-0.4512	0.4103	0.7234	1.6753
	5	-0.3866	-0.3764	-0.3692	0.2218	0.4409	1.0340
	6	-0.3283	-0.3202	-0.3138	0.1571	0.2892	0.6496
	7	-0.2876	-0.2798	-0.2747	0.0984	0.2001	0.4665
3.5	2	-0.9980	-0.9826	-0.9621	6.2831	14.5647	81.5251
	3	-0.6344	-0.6206	-0.6077	1.0010	1.7926	5.1915
	4	-0.4847	-0.4722	-0.4620	0.4342	0.7484	1.6421
	5	-0.3979	-0.3857	-0.3772	0.2309	0.4296	1.0322
	6	-0.3406	-0.3294	-0.3225	0.1691	0.3040	0.6460
	7	-0.2978	-0.2889	-0.2825	0.1156	0.2176	0.4955
4	2	-1.0122	-0.9956	-0.9728	6.8918	14.7036	88.2247
	3	-0.9834	-0.9309	-0.892	3.7529	5.8408	16.5728
	4	-0.4957	-0.4807	-0.4700	0.4514	0.7983	1.8950
	5	-0.4077	-0.3955	-0.3852	0.2694	0.4615	1.0219
	6	-0.3482	-0.3387	-0.3304	0.1817	0.3238	0.6890
	7	-0.3047	-0.2967	-0.2904	0.1033	0.2164	0.5076

Table 12: Upper Record values and BLUE's of μ and σ for$\beta = 1.5 and \delta = 2$

n	Upper record values	μ^*	σ^*
2	1.2920,2.7607	-2.1947	1.9008
3	1.0013,2.2525,2.6529	-0.9837	2.1731
4	1.0193,1.3935,2.1698,3.6395	-1.1536	3.4749
5	0.6795,1.2172,1.6170,3.6803,4.9475	-2.0046	5.6639
6	0.1976,0.2757,0.3639,1.4029,2.3866,2.8864	-1.1791	3.5601
7	1.1294,1.2049,1.4899,1.6223,2.9092,3.0884,3.1585	0.2659	2.6833

Table 13: 95% Confidence interval for μ and σ

n	2	3	4	5	6	7
95%CI forμ (σ known)	(-4.6958,6.4957)	(-2.5048,5.2259)	(-2.1704,2.3257)	(-2.7593,0.5840)	(-1.7188,0.9919)	(-0.1682, 2.1061)
95%CI forμ (σ unknown)	(-51.5954,1.2237)	(-6.7932,0.9874)	(-4.5846,0.9625)	(-4.9499,0.6131)	(-2.3294,0.1565)	(-0.3424,1.1285)
95%CI forσ	(0.3717,21.0192)	(0.3737,4.0533)	(0.5693,3.5899)	(0.9274,4.3103)	(0.5679,2.3356)	(0.4195,1.5819)

Table 14: Predicted records

n	simulated records of size (n-1)	Predicted value
3	2.3061,3.0550	3.5364
4	1.3610,1.4101,2.0967	4.1160
5	0.8188,1.2812,1.3946,2.2052	7.8762
6	0.3431,0.3987,0.9604,1.5604,3.1596	17.6655
7	0.9136,0.9301,1.1862,1.4686,2.2000,2.6492	13.3136
8	1.1294,1.2049,1.4899,1.6223,2.9092,3.0884,3.1585	17.6038

present time, we will be naturally interested in predicting the amount of rainfall or snowfall to be expected when the present record is broken for the first time in future. The best linear unbiased predicted value of the next record can be written as (see Balakrishnan and Chan, 1998).

$$y_{u(n)} = \mu^* + \sigma^* \beta_n$$

where μ^* andσ^* are the BLUE's based on the first (n-1) records and α_n is the n^{th} moment of record values. Prediction of next upper record value is obtained from a simulated data and presented in Table 14.

8 Entropy of Record value distribution

Entropy provides an excellent tool to quantify the amount of information (or uncertainty) contained in a random observation regarding its parent distribution. Shannon's(1948) entropy of an absolutely continuous random variable X with probability density function f(x) is given by

$$H_x[f(x)] = -\int_{-\infty}^{\infty} f(x) ln[f(x)] dx$$

The entropy is always non-negative in the case of a discrete random variable X and is also invariant under one-to- one transformation of X. For a continuous random variable, entropy is not invariant under one-to-one transformation of X and it takes values in $(-\infty, +\infty)$ .The entropy for some commonly used probability distributions have been tabulated by many authors. More recently Ebrahmi et al (2004) have explored the properties of entropy, Kullback - Leibler information and mutual information for order statistics. Now we discuss the entropy for the record values of GLA2E (δ, λ). Let$H_{(R_n)}$ be the entropy of the n^{th} record value. Then by Shakil (2005)

$$H_{(R_n)} = \ln(\Gamma n) - (n-1)\psi(n) - \frac{1}{\Gamma(n)} \int_{-\infty}^{\infty} [-\ln(1-G(x))]^{n-1} g(x) \ln(g(x)) dx \quad (45)$$

where $\int_0^\infty t^{j-1} e{-}tdt = \Gamma(j)$ and $\int_0^\infty t^{j-1} e^{-t} ln(t) dt = \Gamma(j)\psi(j)$ $\psi(j)$ is the digamma function.

For $n = 1$ entropy of the first record value is same as the entropy of parent distribution. Comparison of the entropy of parent distribution and n^{th} record value for $n \geq 2$ is same as comparison of entropy of first record value with entropy of a given n^{th} record value. Since the first observation from the parent distribution is always considered as a record value, entropy of the first non-trivial record value is obtained when $n \geq 2$.

Theorem 8.1. *For GLA2E (δ, β, λ) distribution if $H_{(j)}$ represents the entropy corresponding to j^{th} record, then*

$$H_{(j)} = \ln(\Gamma j) - (j-1)\psi(j) + j - ln(\sigma) + \sum_{i=1}^{\infty} \frac{k^i}{i(i+1)^j} \quad (46)$$

Proof By (45) the entropy of j^{th} record for GLA2E (δ, β, λ) is

$$H_{(j)} = \ln(\Gamma j) - (j-1)\psi(j) - \frac{1}{\Gamma(j)} \int_0^\infty \left[-\ln\left(\frac{\delta}{e^{\beta\lambda x} - \bar{\delta}} \right) \right]^{j-1} v(x) \ln v(x)\ dx$$

where $v(x) = \frac{\delta\lambda e^{\beta\lambda x}}{(e^{\beta\lambda x} - \bar{\lambda})^2}$. By the transformation $t = -\ln \frac{\delta}{e^{\beta\lambda x} - \bar{\delta}}$ and writing $\ln(1 - ke^{-t}) = -\sum_{i=1}^{\infty} \frac{k^i e^{-it}}{i}$ where $k = 1 - \frac{1}{\alpha}$ the result (42) can be easily obtained.

Table 15: Entropy of GLA2E (δ, β, λ)

Record	$\lambda = 0.5$	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$
2	1.6561	0.9630	0.2698	-0.1356
3	2.0250	1.3318	0.6387	0.2332
4	2.2538	1.5606	0.8675	0.4620
5	2.4118	1.7187	1.0255	0.6200
6	2.5296	1.8364	1.1433	0.7378
7	2.6226	1.9295	1.2363	0.8309

Using (46) the entropy of GLA2E (δ, β, λ) for $\delta = \beta = 1.5$ and for various record values and various values of λ are tabulated and presented in Table 15.

9 Conclusion

In this paper, a new family of distributions called Generalized Lehmann Alternative Type II family of distributions is introduced and explored the statistical properties such as probability density function (pdf), hazard rate function (hrf), expressions for cumulative distribution function (cdf), quantile and survival function. Maximum Likelihood function is obtained for estimation of unknown parameters of the new family of distributions. Different special models which include Uniform, Kumaraswamy models are developed for this new family. The probability density function (pdf), cumulative distribution function (cdf) and hazard rate function (hrf) are obtained and plotted the density function for different parameter values. A special model of this family called Generalized Lehmann Alternative Type II Exponential distribution is introduced and studied in detail. The statistical properties such as probability density function (pdf), hazard rate function (hrf), expressions for cumulative distribution function (cdf), quantile and survival function are obtained. The shapes of the density function and hazard rate function are plotted for different parameter values. Method of maximum likelihood estimation is used for estimation of unknown parameters of the new distribution. The new distribution is applied to a real data set to show the effectiveness of the distribution and it is verified that the new model is a better model than the existing exponential model and Marshall-Olkin extended exponential model. A detailed study on the record value theory associated with Generalized Lehmann Alternative Type II Exponential distribution is conducted. Using the mean, variance and covariance of upper record values of the extended model, BLUE's of location and scale parameters are obtained and future records are predicted which has a number of practical uses. The 95% confidence interval for location and scale parameters are also computed. MATLAB programs are developed for this purpose. The result is applied to a real data set to validate the results. Entropy of record values is derived. This result will be useful in characterization of record values based on entropies and a quantification of information contained in each additional record value based on entropy measure.

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Comparison of Machine Learning Techniques for Recommender Systems for Financial Data

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ABSTRACT

Recommender Systems are one of the most successful and widespread application of machine learning technologies in business. These are the software tools used to give suggestions to users on the basis of their requirements. Increase in number of options: be it number of online websites or number of products, it has become difficult for the customer to choose from a wide range of products. Today there is no system available for banks to provide financial advisory services to the customers and offer them relevant products as per their preference before they approach the bank. Like any other industries, financial service rarely has any like, feedback and browsing history to record ratings of services. So it becomes a challenge to build recommender systems for financial services. In this research paper, authors propose a collaborative filtering technique to recommend various products to the customer in order to increase the product per customer (PPC) ratio of bank. The advantage of these recommender systems is that it provides better suggestion to the customer based on his needs/requirements for his/her savings, expenditure and investments.

Key words and Phrases:*Financial Analysis, Recommender systems, Collaborative Filtering, Cosine Similarity, Singular value decomposition.*

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1 Introduction

In today's competitive world, growing customer base and satisfying them is considered the most challenging task. Traditional retail banks with physical branches and high headcounts continue to offer valuable services to consumer but with the increase in digitalisation it becomes a challenge to retain the loyal customers and attract new customers by coping with the new technologies. With digitalisation in financial sector, people expect banks to provide service at their doorstep without being called to the branches. Also with the younger generation i.e. millennial starting with their banking needs, it becomes essential for the financial sector to keep pace with modernisation.

Big data in the banking and financial services have been responsible for helping to create better customer experiences and also help protect businesses. The financial sector and banking institutions can benefit from big data by using that information to customize audience sets by demographic, behaviour, etc. and offer them personalized products (Gigli et al., 2017). Big data facilitates banks and financial institutions to be more specific about product offerings, likely increasing the chance that the right product will be offered to the right person (Amakobe, 2015).

Recommender systems (Carlos et. al 2015, Kumar et. al 2017) are beneficial both to the customer as well as the service providers. These were introduced with the aim of offering products which seems to be more suitable for the customer based on their past behaviour, purchase patterns, financial status and so on. Big Industries like Amazon, Netflix and Facebook uses recommender systems for recommending products, movies and friends to their customers based on the buying behaviour, ratings and browsing history. The existing recommender systems are generally based on ratings, likes, feedback or browsing data. Banking industry has also started embracing digitalisation and has initiated steps to attract customers. Though these techniques are quite common in retail segment like Amazon and across online movie and music industry, this has yet to come strongly into the financial industry. Recommender Systems help stop attrition of the customers by providing quick financial advisory services and help them to find the relevant products at a quick glance be it mortgage, savings account, stocks and bonds, investments or loans. Today there are no such recommendation systems available for financial products to help the customers find their suitable products. Unlike the retail industry, for banking industry, ratings, likes, feedback are not available and so a different methodology had to be derived to find out a single value which represents the ratings of the customers.

Hence, an attempt has been made to propose a recommendation system for service institutions like bank in this paper. The proposed system makes use of the customer demography, income, credit–debit transactions, product holdings etc. to identify customer behaviour and similarity among other customers to provide a very efficient and effective product offering. This will result in improved customer service and customer satisfaction thereby increasing the conversion ratio of the leads and resulting in increased Product per customer (PPC) of the bank.

The organization of the research paper is as follows. The Section 2 provides a detailed literature survey. The proposed framework for Recommender system along with the experimental setup for financial analysis is studied in Section 3. The detailed analysis and the result are presented in Section 4. The last section presents the conclusion and the recommendations.

2 Literature Survey

A recommender system is a technology that is deployed in the environment where items (products, movies, events, articles) are to be recommended to users (customers, visitors, app users, readers) or the opposite (https:/medium.com/recombee-blog/recommender-systems-explained-d98e8221f468). Typically, large number of users and large number of products make it difficult and expensive to know/study every customer's preference and offer the right product or to identify the right customers for each product. Efficient and effective recommender systems are a solution to such situations. Recommender systems provide ratings/preference order to the unrated products based on their past ratings for other products.

Recommender systems (RS) filtering can be categorized into three main approaches. According to Su and Khoshgoftaar (2009) they are

1. Content based filtering: Recommends items by matching attributes with other items that a given user have rated. In content-based recommender systems, a recommendation is based on the relation between the attributes of items that a given user has previously rated and items which the user have not yet rated.

2. Collaborative filtering: Collaborative fultering (CF) based systems propose items based on an analysis of user feedback along with the preferences of similar users. This additional robustness makes CF the most widely used and successful RS method. Recommends items by comparing a given user with a set of users that have rated other items similarly.

3. Hybrid filtering: Recommends items by combining different type of approaches together. This technique overcomes the drawbacks of content based and collaborative filtering technique and improves prediction performance. These have increased complexity and expense for implementation. Also require additional data like unstructured data which is not easily available.

The proposed framework makes use of structured data and user-item similarity based on collaborative filtering technique. Note that, Collaborative filtering (CF) systems have two main approaches for filtering, namely memory-based and model-based. Memory-based collaborative filtering techniques also called neighbourhood-based collaborative filtering includes clustering, user-user and item-item similarity based CF. The methods are based on the correlations or similarity metrics like cosine, Jaccard (Bag et. al 2019) between users and items to produce a preference score that predicts the likelihood of a user acquiring an item in the future and provide corresponding recommendations. User and item-based algorithms are the most common types of memory-based recommendation methods. User-based methods generate recommendations according to the similarities between users, whereas item-based methods compute similarities within a space of items to find strong relationships with items that have already been rated by an active user. These techniques are simple and easy to implement. These techniques use the entire dataset to classify or identify the similarity among customers. The Model-based approach develops a model based on the existing user-item ratings thereby taking a probabilistic approach to calculate the expected value of a user for a particular item. Model based collaborative filtering approach applies statistical method and machine learning technique like Neural Network, Singular Value Decomposition, principal component analysis etc. for mining the rating matrix. In many cases, the ratings matrix is sparse as ratings for every user to item may not exist. These techniques work better with sparse matrix by dimensionality reduction and thereby improve performance. (Su and Khoshgoftaar (2009) and Vijaya Kumar et al. (2014)).

The study presented here compares the performance of two collaborative filtering approaches i.e. memory-based and model-based, using banking industry data. Sarwar et al. (2001) and many others have discussed about both memory-based and model-based techniques in different areas. In this paper, we have implemented both memory based and model based approach for banking dataset to study the performance. The performance of each approach was evaluated using offline testing and user-based testing.

3 Proposed data framework

The proposed framework is to develop recommender systems to offer retail segment products like loans/deposits/ investments to the banking customers as per their life cycle or behavioural requirements. The uses of big data in banking industry are discussed by Amakobe (2015) in the areas of fraud detection, marketing and credit risk management. In India, banks are offering different products to the customers based on their eligibility. But for customer satisfaction, offering right products to the right customers at the right time is more essential. This improves the customer relationship wherein the customer feels that the bank understands their requirements and offers the products well in advance even before the customer reaches out to the bank for his requirements.

In practice, many commercial recommender systems are used to evaluate very large product sets. The user-item matrix used for collaborative filtering will thus be extremely sparse and the performances of the predictions or recommendations of the CF systems are challenged. The data sparsity challenge appears in several situations, specifically, the cold start problem occurs when a new user or item has just entered the system, it is difficult to find similar ones because there is not enough information (in some literature, the cold start problem is also called the new user problem or new item problem). New items cannot be recommended until some users rate it, and new users are unlikely given good recommendations because of the lack of their rating or purchase history (Su and Khoshgoftaar, 2009). In our study also, we face data sparsity issue as more than 70% of the customers hold only Savings bank account with the bank and so the scores for remaining products are not available.

Banking data is very rich and confidential in the sense that it contains all the financial details of the customer like income, loans that he has already availed (for house, education, vehicle etc.), deposits (which shows the liquidity he is holding), investments, spending pattern etc. The data becomes even richer if we make use of unstructured data like transaction mining, browsing history of customers, social media data etc. In this framework we are going to make use of only structured data i.e. the demographics, product holdings, geography, transaction, external bureau data etc. in our study.

Experimental setup: The data of specific set of customers from a bank ABC is considered for the study. Customers getting regular income/salaried in the last six months were included for the study. Data preparation includes missing value imputation and outlier detection. For instance: the birth date of a customer may be a default value or incorrect which gives some unrealistic figure as age or in some cases the birth date may be missing so in these cases, the missing value imputation is done either by using the average value or some other technique depending on the variable. For outlier detection, extreme values are discarded and coerced at 3sigma values. The Data mining process is done using SQL and SPSS Modeller. After data preparation, the biggest challenge is to prepare a user-item rating matrix which will be used as an input for the model. This input matrix has a rating for each customer – product combination.

Ratings/score are the heart of recommendation engines. There are two types of ratings viz. explicit and implicit ratings. Recommender systems that are developed for movie reviews or for e-commerce sites are based on the ratings /feedbacks/likes received from the customers known as explicit ratings. Explicit data is one-action feedback: a single click tells us that a user liked a video or rated a product positively. Explicit data has its own advantages and disadvantages. These are always more valuable to businesses as it is given by the user himself and is clear, unambiguous, and gives a definite picture of the user. But explicit data may also be shallow. Like if a user provides ratings haphazardly only because it was mandatory to provide ratings then the ratings become meaningless (Aggrawal, 2016). For example: Many social media platforms like Face book, twitter etc. have the feature to like a content displayed but the unlike option is not available. Similarly, it is observed that people generally do not provide any positive feedback for any services or applications used but always approach the page for negative feedbacks/complaints. This would result in biased opinion about the product.

One of the challenges of recommender systems in the wider commercial world is that one rarely has explicit ratings data. For example in banking sector there is no concept of providing feedback or rating to a particular product. However, there is often nontrivial information about the interactions, e.g. clicks, purchases, spending pattern etc. Such indirect "ratings" information about user-item interactions is known as implicit feedback. Modelling implicit feedback is a difficult but important problem. The main challenge here is to derive the ratings. The accuracy of the model wholly depends on the ratings and hence at most care has to be given while deriving the implicit ratings. Here we will be dealing with implicit ratings as explicit ratings are not available for banking dataset.

Deriving Score: Implicit ratings have to be derived based on the user behaviour/pattern available. These can be derived either based on some existing document/score card/parameters for a particular item or based on the significant parameters identified through feature selection method. We have performed feature selection for each product separately to identify the significant parameters contributing to each

Figure 1: Word cloud for housing loan parameters

product. Since the behaviour and requirement of each product is different from each other, the weightage of parameters and significance may also differ. For example the parameters found significant for housing loan is shown below in the form of word cloud (Figure 1). Based on the weightage and significance of parameters, a score is derived for each product–customer combination. For instance: If a Customer (C1) has availed a housing loan and his annual income is 16 lakhs whose occupation is State Government employee, EMI is 30% of his net monthly income, then the rating for customer C1 for housing loan will be 35 (= weight assigned to income bracket "¿15 lakhs") +30 (= weight assigned to state government employees) + 10 (weight assigned to EMI bracket 20-30%). So the score for pair C1–housing loan will be 75. **Data Set:** The input matrix used for experiment consists of 1.4 crore rows and 8 columns i.e. approximately 1.4 crores customers and eight products viz. home loan, auto loan, personal loan, pension loan, deposit products like recurring deposit, fixed deposit and investment products like PPF and Mutual Fund. So the input matrix with implicit rankings should ideally be like the table as mentioned below: But since in our case, only 20% of the total customers had availed any other product other than the basic savings bank account, it resulted in a sparse matrix as shown in Table 2.

4 Methodology

In real life scenario, we may be more interested in identifying the top k preferences of a customer rather than estimating the rating that he will give to a particular

Table 1: Input table

User	Product							
	Savings Account	Home Loan	Auto Loan	Personal Loan	Pension Loan	Mutual Fund	PPF	Fixed Deposit
X1	85	56	43	12	32	25	35	43
X2	72	34	23	55	21	44	55	34
—	—	—	—	—	—	—	—	—

Table 2: Sparse matrix table

User	Product							
	Savings Account	Home Loan	Auto Loan	Personal Loan	Pension Loan	Mutual Fund	PPF	Fixed Deposit
X1	32	0	0	43	0	0	0	0
X2	43	0	0	0	0	0	58	43
X3	67	0	45	0	0	0	0	23
—	—	—	—	—	—	—	—	—

product. For instance: the rating a customer may give to a home loan product based on its features like interest rates etc. may be high but it may not be his top preference. In this paper we will be discussing the method to identify the top preferences of a customer based on customer similarity which is also known as the top-k recommendation problem (Aggarwal, 2016). In this paper we will discuss about the collaborative filtering techniques for recommending the top-k products to a customer. Two types of CF techniques viz. memory-based and model-based CF methods are included in this paper.

Memory based CF further includes: k- means clustering technique to segregate the heterogeneous set of customers into homogeneous set thereby identifying the similar set of customers. This method is performed using SPSS Modeller on a system of 64 bit of 48GB RAM and 1 TB storage capacity. This clustering technique gave a silhouette score of 0.6. However, the clusters obtained by this technique were not much differentiable which could be due to sparsity in the data. Thus, clustering technique does not seem to work well with sparse dataset, which is a drawback of this method. The next technique within memory-based CF used here is user-user similarity using Python. This method uses the entire dataset to find similarity among the customers. But due to huge volume of data, the data could not be processed and hence data scalability seems to be a drawback for user-user similarity technique.

Thus major challenges in memory based recommender systems are data sparsity, scalability, diversity etc. Data sparsity leads to the cold start problem i.e. new customers with no purchase history. Also, when number of existing users and items grow tremendously, traditional CF algorithms will suffer serious scalability problems. These problems can degrade the performance of recommendation process.

To achieve better prediction performance and overcome shortcomings of memory-based CF algorithms, model-based CF approaches have been investigated. Model based CF techniques use the rating data to estimate and predicts the top-k preferences (Su and Khoshgoftaar, 2009). Model based CF algorithms include methods such as Bayesian belief nets, Markov Decision Process-based CF, Dimensionality reduction techniques like Singular Value Decomposition (SVD) that are capable of handling problems like data sparsity and scalability.

To overcome the drawbacks of memory-based CF i.e. data sparsity and scalability, Simon Funk's Singular Value Decomposition (SVD) technique which is a model based CF technique is studied in this paper. SVD techniques are dimensionality reducing techniques and hence are known for predicting the ratings for large scale data. After its Success in Netflix Price, Simon Funk's SVD has become a common approach in dealing with huge sparse matrices. Here, the actual rating matrix (A) is decomposed into three matrices as below:

$$A = USI^T,$$

where U is the latent factors matrix of the users, S explains the relationship between the latent factors of the user and item, I is the latent factors matrix of the items. In this case by latent factors we mean characteristics of the user/item.
The resulting dot product calculates the ratings for all user-item pair by minimising the squared error. That is for each rating, error is calculated as

$$E_{ij} = R_{ij} - \hat{R}_{ij}.$$

Using the error values, new values in the User matrix and Item matrix are updated as below. A regularisation parameter is added to the equation to avoid over fitting of the generalised model.

$$U_i = 2\alpha * (A - P) * U_i; \;\; U_i = \text{ith Value in User matrix}$$

$$I_j = 2\alpha * (A - P) * I_j; \;\; I_j = \text{jth value in Item matrix},$$

where α is the learning rate. By multiple iterations, the error is minimized using the gradient descent function. Thus, the nearest approximation is arrived at and the item/service with highest rating is offered as the next best product for the customer.

Our dataset is divided geography-wise into 18 data sets. Each data set consists of approximately 7-8 lakhs data. Only 20% of the customers have availed any other product other than SB. Therefore, the ratings matrix obtained is sparse and hence to deal with data sparsity, Stochastic Gradient Descent (SGD) is used along with SVD popularly as mentioned above to optimize the ratings and thereby minimizing the error. SVD predicts the user-item ratings and hence the error between the actual and the predicted rating value can be calculated. Using the stochastic gradient descent we try to minimize the error through multiple iterations and obtaining the local minima value of error giving the nearest predicted value and increasing the accuracy.

The quality of a recommender system can be decided based on the result of evaluation and interpretation based on business logic. According to Herlocker et al. (2004), metrics evaluating recommendation systems can be broadly classified into the following categories: predictive accuracy metrics, such as Mean Absolute Error (MAE) and its variations; classification accuracy metrics, such as precision, recall, F1-measure, and ROC sensitivity; rank accuracy metrics, such as Pearson's product-moment correlation, Kendall's Tau, Mean Average Precision (MAP), half-life utility, and normalized distance-based performance metric (NDPM). We only introduce the commonly-used CF metrics Mean Absolute Error (MAE) and Root Mean Squared Error (RMSE) here. MAE computes the average of the absolute difference between the predictions and true ratings. The MAE and RMSE are given by

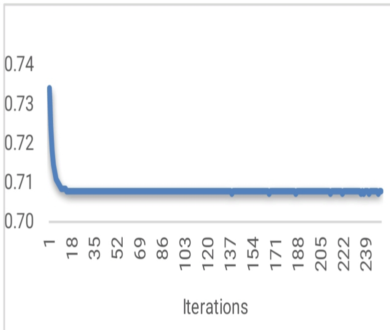
$$MAE = \frac{\sum_{ij} |p_{ij} - r_{ij}|}{n} \tag{4.1}$$

and

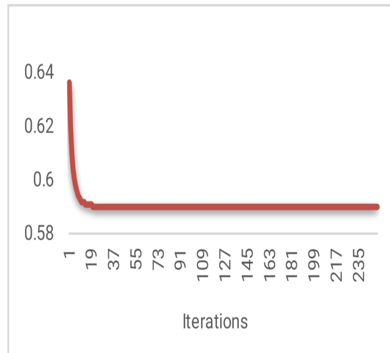
$$RMSE = \sqrt{\frac{1}{n} \sum_{ij} (p_{ij} - r_{ij})^2}, \tag{4.2}$$

where n is the total number of ratings over all users, p_{ij} is the predicted rating for user i on item j, and r_{ij} is the actual rating. The lower the MAE betters the prediction. The RMSE shown in (4.2) amplifies the contributions of the absolute errors between the predictions and the true values. Both MAE and RMSE do not have any upper bounds or lower bounds that justify whether the predictive power or accuracy is good or not. They are only comparable with respect to previous trail or with any other dataset.

Figure 2: Plot of RMSE (a) set-1, (b) set-2



(a)



(b)

Here Funk's SVD is performed with 250 iterations and the learning rate and regularization parameters (Aggrawal, 2016) were finalised based on trial and error methods. The RMSE values become stable after some iteration as shown in Figures 2 (a) and (b) below. The results are also validated as per business logic and it is observed that customers with single product i.e. the cold start problem are also handled well by the algorithm as per their ratings and the suggestions were efficient enough. For example: A person having a tendency of saving is offered investment products like FD, Mutual Funds etc., while a person with existing liability product like home loan may be offered a personal loan. Thus, deriving the score/rating plays a vital role in recommender systems. Therefore, model based CF using Simon Funk's SVD has proved to be providing efficient results.

Table 3 presents the recommendation of the CF algorithm for some select cases. The table may be interpreted as follows: For user-0, the first preference is item number 5 (ie. Recurring deposit), the second preference is item number 7 (mutual fund), and so on. The preference table is shown in Table 4.

Table 3: Recommendation in terms of preferences

User	Item0	Item1	Item2	Item3	Item4	Item5	Item6	Item7	Item8
0	54	75	33	12	53	87	10	66	47
1	42	30	13	53	77	10	64	23	87
2	68	74	21	13	10	34	87	58	49
3	41	81	11	63	30	21	72	35	56
4	29	62	74	20	83	30	12	54	41
5	57	60	31	12	23	20	81	73	46
6	40	79	28	10	84	18	57	31	60
7	22	72	‘62	10	82	17	58	34	40
8	43	55	81	32	14	76	20	27	69
9	41	23	75	20	31	15	86	60	54
10	35	12	26	13	82	66	77	50	45

Table 4: Item preference table

Preference	
Item 0	SB
Item 1	Home Loan
Item 2	Car Loan
Item 3	Personal Loan
Item 4	Pension loan
Item 5	Recurring Deposit
Item 6	Fixed Deposit
Item 7	Mutual Fund
Item 8	PPF

5 Conclusion

The traditional method of offering products to customers without considering their requirement/preference causes irritation among the customers thereby leading to dissatisfaction. Collaborative filtering based recommender systems are the latest techniques that can be used to identify a customer's preference among certain set of products. In this paper, we have discussed memory- based and model-based approaches. In memory-based approach, k-means clustering and user-user similarity were studied. But because of data sparsity and scalability, these approaches did not work for the bank dataset. Whereas, the model-based approach helped in dealing with the data sparsity and scalability and provided efficient results. The proposed framework will help to provide the right product to the right customer. Thereby, improving the customer relationship and satisfaction with the bank. This also helps in reducing the customer attrition rate and increasing the product per customer index. The proposed system though better than the traditional system may have many shortcomings in case of data gaps like unknown salary, inadequate transaction data etc. This can be overcome with the help of Collaborative filtering techniques using big data tools. Capturing the digital footprints of the customer and implementing Hybrid collaborative filtering may also help in improving the prediction power of the models. The newer generation customers expect doorstep banking by ways of digitisation and with increasing levels of expectation they have the tendency of high attrition rate on getting good offers by the competitors. Therefore it is important to foresee the customers' preference and offer the products accordingly to reduce the attrition rate. Thus Recommender systems can be of huge benefit for the financial industry by using it wisely to attract customers.

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Measure of Slope Rotatability for Second Order Response Surface Designs Under Tri-Diagonal Correlation Error Structure Using Central Composite Designs

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ABSTRACT

In the design of experiments for estimating the slope of the response surface, slope rotatability is a desirable property. In this paper, measure of slope rotatability for second order response surface designs using central composite designs under tri-diagonal correlation error structure is suggested and illustrated with examples.

Key words and Phrases:*Response surface design, slope-rotatability, tri-diagonal correlation error structure, central composite designs, weak slope rotatability region.*
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1 Introduction

Response surface methodology is a collection of mathematical and statistical techniques useful for analyzing problems where several independent variables influence

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a dependent variable. The independent variables are often called the input or explanatory variables and the dependent variable is often the response variable. An important step in development of response surface designs was the introduction of rotatable designs by Box and Hunter (1957). Das and Narasimham (1962) constructed rotatable designs using balanced incomplete block designs (BIBD). The study of rotatable designs mainly emphasized on the estimation of absolute response. Estimation of response at two different points in the factor space will often be of great importance. If differences at two points close together, estimation of local slope (rate of change) of the response is of interest. Hader and Park (1978) extended the notion of rotatability to slope rotatability for the case of second order models. In view of slope rotatability of response surface methodology, a good estimation of derivatives of the response function is more important than estimation of mean response. Estimation of slopes occurs frequently in practical situations. For instance, there are cases in which we want to estimate rate of reaction in chemical experiment, rate of change in the yield of a crop to various fertilizer doses, rate of disintegration of radioactive material in animal etc. (cf. Park 1987). Victorbabu and Narasimham (1991) studied second order slope rotatable designs (SOSRD) using BIBD. Victorbabu (2007) suggested a review on SOSRD. To access the degree of slope rotatability Park and Kim (1992) introduced a measure for second order response surface designs. Victorbabu and surekha (2011) studied measure of slope rotatability for second order response surface designs using central composite designs (CCD).

Many authors have studied rotatable designs and slope rotatable designs assuming errors to be uncorrelated and homoscedastic. However, it is not uncommon to come across practical situations when the errors are correlated, violating the usual assumptions. Das (1997, 2003a) introduced and studied robust second order rotatable designs. Das (2003b) introduced slope rotatability with correlated errors and gave conditions for the different variance-covariance error structures. To access the degree of slope rotatability for correlated errors a new measure for second order response surface designs was introduced by Das and Park (2009). Rajyalakshmi and Victorbabu (2014, 15, 18, 19) studied SOSRD under tri-diagonal structure of errors using CCD, pairwise balanced designs, symmetrical unequal block arrangements (SUBA) with two unequal block sizes and BIBD respectively. Sulochana and Victorbabu (2020a, 2020b) studied SOSRD under tri-diagonal correlated structure of errors using a pair of SUBA with two unequal block sizes and a pair of BIBD respectively. Sulochana and Victorbabu (2020c, 2021a, 21b, 21c) studied measure of slope rotatability for second order response surface designs under intra-class correlated structure of errors using PBD,CCD, BIBD and SUBA with two unequal block sizes respectively.

In this paper, following the works of Park and Kim (1992), Das (2003a, 2003b, 2014), Das and Park (2009), Surekha and Victorbabu (2011), Rajyalakshmi and Victorbabu (2014), the measure of slope-rotatability for second order response surface designs with tri-diagonal correlation error structure using CCD for $\rho(0.9 \leq \rho \leq 0.9)$ for $2 \leq v \leq 8$ (v number of factors) is suggested.

2 Preliminaries

2.1 Tri-diagonal correlation structure

The tri-diagonal structure of errors arises when the variance is same (σ^2) and the correlation between any two errors having lag n is ρ, and 0 (zero) otherwise. The tri-diagonal error structure with $2n$ observations is given below. (cf. Das (2014) p.30).

$$W = \left\{ D(e) = \sigma^2 \left[\begin{bmatrix} I_n & I_n \\ I_n & I_n \end{bmatrix} \times \frac{1+\rho}{2} + \begin{bmatrix} I_n & -I_n \\ -I_n & I_n \end{bmatrix} \times \frac{1-\rho}{2} \right] = W_{2n\times 2n}(\rho), \text{say} \right\}$$

$$W^{-1}_{2n\times 2n}(\rho) = \left(\sigma^2\right)^{-1} \left[\begin{bmatrix} I_n & I_n \\ I_n & I_n \end{bmatrix} \times \frac{1}{2(1+\rho)} + \begin{bmatrix} I_n & -I_n \\ -I_n & I_n \end{bmatrix} \times \frac{1}{2(1+\rho} \right]$$

2.2 Conditions of slope rotatability for second order response surface designs with tridiagonal correlation error structure (Das 2003a, 2003b, 2014)

A second order response surface design $D = (X_{ui})$ for fitting,

$$Y_u(X) = b_0 + \sum_{i=1}^{v} b_i X_{ui} + \sum_{i=1}^{v} b_{ii} X_{ui}^2 + \sum_{i \le j = 1}^{n} b_{ij} X_{ui} X_{uj} + e_u;\ 1 \le u \le 2n \qquad (2.1)$$

where X_{ui} denotes the level the ith factor $(i = 1, 2, \ldots v)$ in the u^{th} run $(u = 1, 2, \ldots 2n)$ of the experiment, e_u's are correlated random errors, is said to be a SOSRD under tri-diagonal correlated structure of errors, if the variance of the estimate of first order partial derivative of $Y_u(X_{u1}, X_{u2}, X_{u3} \ldots, X_{uv})$ with respect to each independent variable (X_i) is only a function of the distance $\left(d^2 = \sum_{i=1}^{v} X_i^2\right)$ of the point $(X_{u1}, X_{u2}, X_{u3} \ldots, X_{uv})$ from the origin (center of the design). i.e, $V\left(\frac{\partial \hat{Y}_u}{\partial X_i}\right) = h\left(d^2\right)$. Such a spherical variance function $h(d^2)$ for estimation of slopes in the second order response surface is achieved if the design points satisfy the following conditions (see, Das 2003a, 2003b and 2014, Rajyalakshmi and Victorbabu 2014, 2015, 2018, 2019).

Following Box and Hunter (1957), Hader and Park (1978), Victorbabu and Narasimham (1991a), Das (2003a, 2003b and 2014), Rajyalakshmi and Victorbabu (2014, 2015, 2018, 2019) the general conditions for second order slope rotatability under the tri-diagonal correlated structure of errors can be obtained as follows. To simplify the fit of the second order polynomial from design points D through the method of least squares, we impose the following simple symmetry conditions on D to facilitate easy solutions of the normal equations. (cf. Das, 2014, p. 67, 112-114).

(I)

$(i)\ \sum_{u=1}^{2} nX_{uj} = 0;\ 1 \leq j \leq v,$

$(ii) \sum_{u=1}^{2} nX_{uj}X_{ui} = 0;\ 1 \leq j < 1 \leq v,$

$(iii) \sum_{u=1}^{2} nX_{ui}X_{uj} - \rho\left\{\sum_{u=1}^{n} X_{(n+u)i}X_{uj} + \sum_{u=1}^{n} X_{ui}X_{(n+u)j}\right\} = 0,\ 1 \leq i \neq j \leq v,$

$(iv)\ \sum_{u=1}^{2} nX_{ui}^{2}X_{uj} - \rho\left\{\sum_{u=1}^{n} X_{(n+u)i}^{2}X_{uj} + \sum_{u=1}^{n} X_{ui}^{2}X_{(n+u)j}\right\} = 0,\ 1 \leq i \neq j \leq v,$

$$(v)\ \sum_{u=1}^{2n} X_{ui}X_{uj}X_{ul} - \rho\left\{\sum_{u=1}^{n} X_{(n+u)i}X_{(n+u)j}X_{ul} + \sum_{u=1}^{n} X_{ui}X_{uj}X_{(n+u)l}\right\} = 0, \tag{2.2}$$

$, 1 \leq i \neq j \leq v, 1 \leq l \leq v,$

$$(vi)\ \sum_{u=1}^{2n} X_{ui}^{2}X_{uj}X_{ul} - \rho\left\{\sum_{u=1}^{n} X_{(n+u)i}^{2}X_{(n+u)j}X_{ul} + \sum_{u=1}^{n} X_{ui}^{2}X_{uj}X_{(n+u)l}\right\} = 0, \tag{2.3}$$

$1 \leq i \leq v, 1 \leq j < l < v,$

$$(vii)\ \sum_{u=1}^{2n} X_{ui}X_{uj}X_{ul}X_{ut} - \rho\left\{\sum_{u=1}^{n} X_{(n+u)i}X_{(n+u)j}X_{ul}X_{ut} + \sum_{u=1}^{n} X_{ui}X_{uj}X_{(n+u)l}X_{(n+u)t}\right\} = 0, \tag{2.4}$$

$1 \leq i < l < j \leq v, 1 < t \leq v; (i,j) \neq (1,t) \neq j \leq v, 1 \leq l \leq v,$

$$\sum_{u=1}^{2n} X_{ui}^{2} = \text{a constant} = 2n\lambda_2,\ \text{for all } i, \tag{2.5}$$

$$\sum_{u=1}^{2n} X_{ui}^{4} = \text{constant} = c2n\lambda_2,\ \text{for all } i, \tag{2.6}$$

$$\sum_{u=1}^{2n} X_{ui}^{2}X_{uj}^{2} = \text{constant} = 2n\lambda_4,\ \text{for all values } i \neq j \tag{2.7}$$

$$\sum_{u=1}^{2} nX_{ui}^{4} = c\sum_{u=1}^{2} nX_{ui}^{2}X_{uj}^{2} \tag{2.8}$$

Using (2.3), (2.4) and (2.5) the design parameters of the tri-diagonal correlated structure are as follows:

(II)

$(i)(1-\rho)\left\{\sigma^2(1-\rho^2)\right\}^{-1}\sum_{u=1}^{2n} X_{ui}^2 = \frac{2n\lambda_2(1-\rho)}{\sigma^2(1-\rho^2)}(>0), 1 \le j \le v,$

$(ii)\left\{\sigma^2(1-\rho^2)\right\}^{-1}\left[\sum_{u=1}^{2n} X_{ui}^2 - 2\rho\sum_{u=1}^{n} X_{ui}X_{(n+u)i}\right] = \frac{2n\lambda_2}{\sigma^2(1-\rho^2)}(>0), 1 \le i \le v,$

$(iii)\left\{\sigma^2(1-\rho^2)\right\}^{-1}\left[\sum_{u=1}^{2n} X_{ui}^4 - 2\rho\sum_{u=1}^{n} X_{ui}^2 X_{(n+u)i}^2\right] = c\left[\frac{2n\lambda_4}{\sigma^2(1-\rho^2)}\right](>0), 1 \le i \le v,$

$(iv)\left\{\sigma^2(1-\rho^2)\right\}^{-1}\left[\sum_{u=1}^{2n} X_{ui}^2 X_{uj}^2 - \rho\left[\sum_{u=1}^{n} X_{(n+u)i}^2 X_{uj}^2 - \sum_{u=1}^{n} X_{ui}^2 X_{(n+u)j}^2\right]\right] =$
$\left[\frac{2n\lambda_4}{\sigma^2(1-\rho^2)}\right](>0), 1 \le i \ne j \le v,$

$(v)\left\{\sigma^2(1-\rho^2)\right\}^{-1}\left[\sum_{u=1}^{2n} X_{ui}^2 X_{uj}^2 - 2\rho\left[\sum_{u=1}^{n} X_{ui}X_{uj}X_{(n+u)i}X_{(n+u)j}\right]\right] = \left[\frac{2n\lambda_4}{\sigma^2(1-\rho^2)}\right](>0),$
$1 \le i < j \le v,$

From (II) of (iii), (iv) and (v)

$$\left\{\sigma^2(1-\rho^2)\right\}^{-1} \quad \left[\sum_{u=1}^{2n} X_{ui}^4 - 1\rho\sum_{u=1}^{n} X_{ui}^2 X_{(n+u)i}^2\right] = 2\left(\left\{\sigma^2(1-\rho^2)\right\}^{-1}\right)$$
$$\left[\sum_{u=1}^{2n} X_{ui}^2 X_{uj}^2 - 2\rho\left[\sum_{u=1}^{n} X_{ui}X_{uj}X_{(n+u)i}X_{(n+u)j}\right]\right]$$
$$+\ \left\{\sigma^2(1-\rho^2)\right\}^{-1}$$
$$\left[\sum_{u=1}^{2n} X_{ui}^2 X_{uj}^2 - \rho\left[\sum_{u=1}^{n} X_{(n+u)i}^2 X_{uj}^2 - \sum_{u=1}^{n} X_{ui}^2 X_{(n+u)j}^2\right]\right]$$

which implies to

$$c\left(\frac{2n\lambda_4}{\sigma^2(1-\rho^2)}\right) = \eta\left[2\left(\frac{2n\lambda_4}{\sigma^2(1-\rho^2)}\right) + \left(\frac{2n\lambda_4}{\sigma^2(1-\rho^2)}\right)\right]$$

where $c = 3\eta$, $n = 2n$, η, λ_2 and λ_4 are constants. The summation is over the designs points, and ρ is the correlation coefficient.

The variance and covariances of the estimated parameters under the tri-diagonal correlated structure of errors are as follows:

$$V\left(\hat{b}_0\right) = \frac{\sigma^2\lambda_4(c+v-1)(1+\rho)}{2n\Delta} \tag{2.9}$$

$$V\left(\hat{b}_i\right) = \frac{\sigma^2(1-\rho^2)}{2n\lambda_2} \tag{2.10}$$

$$V\left(\hat{b}_{ij}\right) = \frac{\sigma^2(1-\rho^2)}{2n\lambda_4} \tag{2.11}$$

$$V\left(\hat{b}_{ii}\right) = \frac{\sigma^2(1-\rho^2)\left[\lambda_4(c+v-2)-(v-1)\lambda_2^2(1-\rho)\right]}{(c-1)(2n)\lambda_4\Delta} \tag{2.12}$$

$$Cov\left(\hat{b}_0, \hat{b}_{ii}\right) = -\frac{\sigma^2\lambda_2^2(1-\rho^2)}{2n\Delta} \tag{2.13}$$

$$Cov\left(\hat{b}_{ii}, \hat{b}_{ij}\right) = \frac{\sigma^2(1-\rho^2)\left[\lambda_2^2(1-\rho)-\lambda_4\right]}{(c-1)(2n)\lambda_4\Delta} \tag{2.14}$$

where $\Delta = \left[\lambda_4(c+v-1) - v\lambda_2^2(1-\rho)\right]$ and the other covariances are zero.

An inspection of the variance of $\hat{b}_0$ shows that a necessary condition for the existence of a nonsingular second order slope rotatable design with tri-diagonal correlated structure is

$$\left[\lambda_4(c+v-1) - v\lambda_2^2(1-\rho)\right] > 0 \tag{2.15}$$

$$\frac{\lambda_4}{\lambda_2^2} > \frac{v(1-\rho)}{c+v-1} \qquad \text{(non} - \text{singularity condition)} \tag{2.16}$$

If the non-singularity condition (2.14) exists, then only the design exists.

For the second model

$$\frac{\partial \hat{Y}_u}{\partial X_i} = \hat{b}_i + 2\hat{b}_{ii}X_i + \sum_{i=j\neq i}^{v} \hat{b}_{ij}X_j,$$

$$V\left(\frac{\partial \hat{Y}_u}{\partial X_i}\right) = V\left(\hat{b}_i\right) + 4X_i^2 V\left(\hat{b}_{ii}\right) + \sum_{i=j\neq i}^{v} X_j^2 V\left(\hat{b}_{ij}\right) \tag{2.17}$$

The condition for right hand side of equation (2.15) to be a function of ($d^2 = \sum_{i=1}^{v} X_i^2$) alone (for slope rotatability) is clearly,

$$V\left(\hat{b}_{ii}\right) = \frac{1}{4}V\left(\hat{b}_{ii}\right) \tag{2.18}$$

Equation (2.16) leads to condition,

$$\frac{cN\lambda_4}{(1-\rho^2)(1+\rho)}\left[4N-(c+v-2)N+v\left(\frac{N\lambda_2^2(1-\rho)}{\lambda_4}\right)\right]+\frac{N^2\lambda_4}{(1-\rho^2)(1+\rho)}[5v-9]-N^2\lambda_2^2\left[\frac{5v-4}{(1+\rho)^2}\right]=0 \tag{2.19}$$

where $N = 2n$. Simplifying (2.17) gives rise,

$$\lambda_4[v(5-c)-(c-3)^2]+\lambda_2^2[v(c-5)+4](1-\rho)=0 \tag{2.20}$$

For $\rho = 0$, equation (2.18) reduces to

$$\lambda_4[v(5-c)-(c-3)^2]+\lambda_2^2[v(c-5)+4]=0 \tag{2.21}$$

Equation (2.19) is similar to the SOSRD condition of Victorbabu and Narasimham (1991a).

Therefore, equations (2.2) to (2.12), (2.14) to (2.18) give a set of conditions for SOSRD under tri-diagonal correlated structure of errors for any general second order response surface design.

On simplification of (2.15) using (2.7) to (2.12) and (2.16), we have,

$$\begin{aligned}
V\left(\frac{\partial \hat{Y}_u}{\partial X_i}\right) &= V(\hat{b}_i)+4X_i^2\frac{V\left(\hat{b}_{ij}\right)}{4}+\sum_{i=1,j\neq i}^{v}X_j^2V\left(\hat{b}_{ij}\right)\\
V\left(\frac{\partial \hat{Y}_u}{\partial X_i}\right) &= V(\hat{b}_i)+\sum_{i=1}^{v}X_i^2V\left(\hat{b}_ij\right)\\
V\left(\frac{\partial \hat{Y}_u}{\partial X_i}\right) &= V(\hat{b}_i)V\left(\hat{b}_{ij}\right)d^2
\end{aligned}$$

where $d^2=\sum_{i=1}^{v}X_i^2$ and $V\left(\hat{b}_i\right)$, $V\left(\hat{b}_ij\right)$ are stated in (2.8) and (2.9). Further, we have

$$V\left(\frac{\partial \hat{Y}_u}{\partial X_i}\right)=\frac{1-\rho^2}{N}\left(\frac{1}{\lambda_2}+\frac{d^2}{\lambda_4}\right)\sigma^2 \tag{2.22}$$

where $N = 2n$.

2.3 Slope rotatability for second order response surface designs with tri-diagonal correlation error structure using CCD (Rajyalakshmi and Victorbabu (2014))

Following the works of Hader and Park (1978), Victorbabu and Narasimham (1991), Das (2003a, 03b, 2014), Rajyalakshmi and Victorbabu (2014), the method construction of slope rotatability for second order response surface designs with tri-diagonal correlation error structure using CCD is given below.

Central composite designs are obtained by adding suitable factorial combinations $(\pm 1, \pm 1, \ldots, \pm 1)$ to those obtained from $2^{t(v)}$ fractional (or a suitable fractional replicate of 2^v in which no interaction less than five factors is confounded). The $2v$ additional fractional combinations in CCD are $(\pm\alpha, 0, \ldots, 0)$, $(0, \pm\alpha, 0 \ldots, 0), \ldots (0, 0, \ldots, \pm\alpha)$ and n_0 central points $(0, 0, ..., 0)$ if necessary. The total number of factorial combinations in the design can be written as $N = F + T$. Here F is total number of fractional points. i.e., $F = 2^{t(v)}$ and $T = 2v + n_0$.

Here we consider a slope rotatable central composite designs of Hader and Park (1978) having $'n'(n = F + 2v)$ non-central design points involving v-factors. The set of $'n'$ - non central design points are extended to $2n$ design points by adding $'n'(n_0 = n)$ central points just below or above the $'n'$ non-central design points. Hence $2n(= N)$ be the total number of design points of the slope rotatability for second order response surface designs with tri-diagonal correlation error structure using CCD.

Result (2.1): The design points $(\pm 1, \pm 1, \ldots, \pm 1)F \cup (\pm\alpha, \ldots, 0)2^1 \cup n_0$ will give a v-dimensional SOSRD with tri-diagonal correlation error structure using CCD in design points $N = F + T$, where α^2 is positive real root of the fourth degree polynomial equation,

$$[(8v(1-\rho) - 4N)]\alpha^8 + [8Fv(1-\rho)]\alpha^6 + [(2F(4-v)N + 2F^2v(1-\rho) + 16F(1-v))]\alpha^4$$
$$+[\big(16F^2(1-v)(1-\rho)\big)]\alpha^2 + [\big(4F^2(v-1)N + 4F^3(1-v)(1-\rho)\big)] = 0$$

Note: Values of SOSRD under tri-diagonal correlation error structure using CCD can be obtained by solving the above equation.

3 Measure of second order slope rotatability for correlated structure of errors (Das and Park, 2009))

Following Das and Park (2009), equations (2.2) to (2.18) give necessary and sufficient conditions for a measure for any second order response surface designs with correlated errors. Further we have,

$$\begin{aligned}
&V(b_i) \text{ equal for all } i,\\
&V(b_{ii}) \text{ equal for all } i\\
&V(b_{ij}) \text{ equal for all } i,j;\ \text{where } i \neq j\\
&Cov(b_i,b_{ii}) = Cov(b_i,b_{ij}) = Cov(b_{ii},b_{ij}) = Cov(b_{ij},b_{il}) = 0 \text{ for all } i \neq j \neq l \text{ and for all } \rho
\end{aligned} \tag{3.1}$$

Das and Park (2009) proposed that, if the conditions in (2.2) to (2.18) and (3.1) are met, $M_v(D)$ is the proposed measure of slope rotatability for second order response surface designs for any correlated error structure.

$$M_v(D) = \frac{1}{1+Q_v(D)}$$

$$\begin{aligned}
\text{where } 2(v-1)\sigma^4 Q_v(D) \quad = \quad & (v+2)(v+4)\sum_{i=1}^{v}\left[(V(b_i)-\bar{V}) + \frac{a_i-\bar{a}}{v+2}\right]^2\\
& + \frac{4}{v(v+2)}\sum_{i=1}^{v}(a_i-\bar{a})^2 + 2\sum_{i=1}^{v}\left[\left(4V(b_{ii})\frac{a_i}{v}\right)^2 + \sum_{i=1\neq j}^{v}\left(V(b_{ij})\frac{a_i}{v}\right)^2\right]\\
& 4(v+4)\left[4Cov(b_i,b_{ii})^2 + \sum_{j=1;j\neq i}^{v} Cov(b_i,b_{ij})^2\right]\\
& 4\sum_{l=1}^{v}\left(4\sum_{j=1;j\neq i}^{v} Cov(b_{ii},b_{ij})^2\right) + \sum_{j<l}\sum_{l\neq i}^{v} Cov(b_{ij},b_{lj})^2
\end{aligned} \tag{3.2}$$

Here $\bar{V} = \frac{1}{v}\sum_{i=1}^{v} V(b_i)$, $a_i = 4V(b_{ii}) + \sum_{j=1;j\neq i}^{v} V(b_{ij})(1 \leq i \leq v)$ and $\bar{a} = \frac{1}{v}\sum_{i=1}^{v} a_i$.

It can be easily shown that $Q_v(D)$in equation (3.2) becomes zero for all values n, if and only if the conditions in equations (3.1) hold. Further, it is simplified to

$$Q_v(D) = \frac{1}{\sigma^4}\left[4V(b_{ii}) - V(b_{ij})\right]^2. \tag{3.3}$$

Note that $0 \leq M_v(D) \leq 1$, and it can be easily shown that $M_v(D)$ is one if and only if the design is slope rotatable with any correlated error structure for all values of ρ, and $M_v(D)$ approaches to zero as the design $'D'$ deviates from the slope-rotatability under specified correlated error structure.

4 Measure of slope rotatability for second order response surface designs with tri-diagonal correlation error structure using central composite designs

Following Park and Kim (1992), Das and Park (2009), Surekha and Victorbabu (2011), Rajyalakshmi and Victorbabu (2014) the proposed measure of slope-rotatability for second order response surface designs with tri-diagonal correlation error structure using CCD is given below.

This well-known type of design consists of $2^{t(v)}$ factorial points $(\pm 1, \pm 1, \ldots, \pm 1)$, $2v$ axial points of the form $(\pm\alpha, 0, \ldots, 0)$ and a center point $(0, 0, ..., 0)$ may be replicated n_0 times if necessary. The total number of factorial combinations in the design can be written as $N = F + T$. Here F is total number of fractional points. i.e., $F = 2^{t(v)}$ and $T = 2v + n_0$.

Here we consider a slope rotatable central composite designs of Hader and Park (1978) having $n(n = F + 2v)$ non-central design points involving v-factors. The set of n- non central design points are extended to $2n$ design points by adding $n(n_0 = n)$ central points just below or above the n non-central design points. Hence $2n(= N)$ be the total number of design points of the slope rotatability for second order response surface designs with tri-diagonal correlation error structure using CCD.

The design points $(\pm 1, \pm 1, ..., \pm 1)F \cup (\pm\alpha, ..., 0)2^1 \cup n_0$, will give measure of slope rotatability for second order response surface designs withtri-diagonal correlation error structure using CCD. Here we have (2.2) are true. Further, from (2.3), (2.4), and (2.5), we have,

$$\begin{aligned}
&(I)\sum_{u=1}^{2n} X_{ui}^2 = F + 2\alpha^2 = 2n\lambda_2 \\
&(II)\sum_{u=1}^{2n} X_{ui}^4 = F + 2\alpha^4 = c2n\lambda_4 \\
&(III)\sum_{u=1}^{2n} X_{ui}^2 X_{uj}^2 = F = 2n\lambda_4
\end{aligned} \tag{4.1}$$

Measure of slope rotatability of second order response surface designs with tri-diagonal correlation error structure using CCD can be obtained by

$$\begin{aligned}
M_v(D) &= \frac{1}{1+Q_v(D)} \\
1+Q_v(D) &= \frac{1}{\sigma^4}[4V(b_{ii}) - V(b_{jj})]^2 \\
&= \frac{1}{\sigma^4}[4G - \frac{(1-\rho^2)\sigma^2}{F}]^2
\end{aligned} \tag{4.2}$$

where $G = V(b_{ii}) = (1-\rho^2)\sigma^2 \left[\frac{(v-1)FT - F(v-1)\rho - 4(v-1)F\alpha^2 + 2[N-2(v-1)]\alpha^4}{2\alpha^4[vFT - Fv\rho - 4vF\alpha^2 + 2[N-2v]\alpha^4]}\right]$

If $M_v(D)$ is one if and only if the design ‘D ’ is slope rotatable withtri-diagonal correlationerror structure using CCD for all values of ρ and $M_v(D)$ approaches to zero as the design ‘D ’deviates from the slope-rotatability withtri-diagonal correlation error structure using CCD.

Example: We illustrate the measure of slope-rotatability for second order response surface designs withtri-diagonal correlated structure of errors with the help of CCD for v=2 factors.

The design points $(\pm1, \pm1, \pm1)2^2 \cup (\pm\alpha, ..., 0)2^1 \cup n_0$, will give slope rotatability for second order response surface designs withtri-diagonal correlation error structure in $N = 16$ design points for 2 factors. From equations in (4.1), we have

$$\begin{aligned}
&(I)\sum X_{ui}^2 = 4 + 2\alpha^2 = N\lambda_2 \\
&(II)\sum X_{ui}^4 = 4 + 2\alpha^4 = cN\lambda_4 \\
&(III)\sum X_{ui}^2 X_{uj}^2 = 4 = N\lambda_4
\end{aligned} \tag{4.3}$$

From (I), (II) and (III) of (4.3), we get $\lambda_2 = \frac{4+2\alpha^2}{16}$, $\lambda_4 = \frac{4}{16}$ and $c = \frac{4+2\alpha^4}{4}$ and Substituting λ_2,λ_4 and c in (3.5) and on simplification, we get the following biquadratic equation in α^2.

$$[16(1-\rho)-64]a^8+64(1-\rho)a^6+[256]a^4-256(1-\rho)a^2+[1024-256(1-\rho)] = 0 \quad (4.4)$$

Equation (4.4) has only one positive real root for each value of ρ. This can be alternatively written directly from result (2.1). Solving (4.4), we get $\alpha = 1.7254$. From (4.2) we get $Q_v(D) = 0$,$M_v(D) = 1$ for value of $\rho = 0.1$.

Suppose if we take $\alpha = 1.3$ instead of taking $\alpha = 1.7254$ for 2 factors we get $Q_v(D) = 0.1519$ then $M_v(D) = 0.8790(taking\rho = 0.1)$. Here $M_v(D)$ deviates from slope rotatability for second order response surface designs withtri-diagonal correlation error structure.

4.1 Weak slope rotatability region for correlated errors(cf. Das and Park (2009)

Following Das and Park (2009), we also find weak slope rotatability region (WSRR) for second order response surface designs with tri-diagonal correlation error structure using CCD.

$M_v(D) \geq k$

$M_v(D)$ involves the correlation parameter $\rho\epsilon W$ and as such, $M_v(D) \geq k$ for all ρ is too strong to be met. On the other hand, for a given v, we can find range of values of ρ for which Das and Park (2009) call this range as the weak slope rotatability region $(WSRR(R_{D(k)}(\rho)))$ of the design 'D'. Naturally, the desirability of using 'D' will rest on the wide nature of $(WSRR(R_{D(k)}(\rho)))$ along with its strength k. Generally, we would require 'v' to be very high say, around 0.95 (cf. Das and Park (2009)).

Table 1 and 2, gives the values of $M_v(D)$ and weak slope rotatability region $(WSRR(R_{D(k)}(\rho)))$ for second order slope rotatable designs with tri-diagonal correlationerror structure using CCD for $\rho(-0.9 \leq \rho \leq 0.9)$ and $2 \leq v \leq 8$ (v number of factors) respectively.

Table 1: Measure of slope rotatability for second order response surface designs ($M_V(D)$) with tri-diagonal correlation error structure using CCD $\rho(-0.9 \leq \rho \leq 0.9)$ and for $2 \leq v \leq 4$

v = 2, n=8, 2n=N= 16									
ρ	α								
	1	1.3	1.6	1.9	2.2	2.5	2.8	3.1	α^*
-0.9	0.7596	0.9009	0.9617	0.9964	0.9995	0.9963	0.9935	0.9917	2.1756
-0.8	0.6353	0.8578	0.9559	0.9974	0.9982	0.9919	0.9872	0.9841	2.105
-0.7	0.5614	0.8385	0.9593	0.9989	0.996	0.9875	0.9812	0.9771	2.0411
-0.6	0.5141	0.8313	0.9652	0.9998	0.9934	0.9831	0.9757	0.971	1.9836
-0.5	0.4831	0.8308	0.9713	0.9999	0.9907	0.9789	0.9708	0.9657	1.9321
-0.4	0.4628	0.8344	0.9769	0.9999	0.9879	0.9753	0.9667	0.9613	1.8863
-0.3	0.4503	0.8406	0.9819	0.9989	0.9853	0.9721	0.9633	0.9577	1.8457
-0.2	0.4439	0.8486	0.9859	0.9979	0.9831	0.9696	0.9607	0.9551	1.8097
-0.1	0.4428	0.8579	0.9894	0.9969	0.9812	0.9677	0.9589	0.9533	1.7781
0	0.4464	0.8681	0.9921	0.9959	0.9798	0.9666	0.9579	0.9525	1.7501
0.1	0.4546	0.8791	0.9942	0.9949	0.9789	0.9661	0.9579	0.9527	1.7254
0.2	0.4676	0.8906	0.9959	0.9942	0.9785	0.9665	0.9587	0.9538	1.7036
0.3	0.4861	0.9027	0.9972	0.9937	0.9788	0.9676	0.9604	0.9558	1.6843
0.4	0.5109	0.9153	0.9982	0.9934	0.9797	0.9695	0.9629	0.9589	1.6671
0.5	0.5438	0.9283	0.9989	0.9935	0.9813	0.9723	0.9666	0.9629	1.6517
0.6	0.5869	0.9417	0.9993	0.9939	0.9835	0.9759	0.9711	0.9681	1.6379
0.7	0.6446	0.9556	0.9997	0.9948	0.9865	0.9805	0.9766	0.9743	1.6255
0.8	0.7229	0.9699	0.9999	0.9961	0.9902	0.986	0.9833	0.9816	1.6143
0.9	0.8339	0.9847	0.9999	0.9978	0.9947	0.9925	0.9911	0.9902	1.6042

Table 1 continued

v = 3, n=14, 2n=N= 28									
ρ	α								
	1	1.3	1.6	1.9	2.2	2.5	2.8	3.1	α^*
-0.9	0.7529	0.9595	0.9904	0.9976	0.9999	0.9998	0.9991	0.9985	2.4616
-0.8	0.6175	0.928	0.9843	0.997	0.9999	0.9994	0.9981	0.9969	2.3772
-0.7	0.5335	0.9035	0.9804	0.9969	0.9999	0.9988	0.997	0.9955	2.3028
-0.6	0.4776	0.8845	0.9779	0.9972	0.9999	0.9982	0.9959	0.9942	2.2386
-0.5	0.4391	0.8699	0.9766	0.9976	0.9999	0.9976	0.9951	0.9931	2.1843
-0.4	0.4122	0.8593	0.9759	0.9979	0.9997	0.9971	0.9942	0.9921	2.139
-0.3	0.3937	0.8519	0.9758	0.9983	0.9995	0.9965	0.9935	0.9913	2.1015
-0.2	0.3816	0.8474	0.9763	0.9986	0.9993	0.9961	0.9929	0.9907	2.0703
-0.1	0.3751	0.8456	0.9769	0.9989	0.9991	0.9957	0.9926	0.9903	2.0444
0	0.3733	0.8464	0.978	0.9991	0.9989	0.9955	0.9923	0.99	2.0226
0.1	0.3763	0.8496	0.9793	0.9993	0.9988	0.9953	0.9922	0.99	2.0226
0.2	0.3841	0.8552	0.9809	0.9995	0.9987	0.9953	0.9924	0.9902	1.9884
0.3	0.3975	0.8632	0.9826	0.9996	0.9987	0.9954	0.9926	0.9907	1.9749
0.4	0.4173	0.8738	0.9846	0.9997	0.9987	0.9956	0.9931	0.9913	1.9632
0.5	0.4457	0.8869	0.9867	0.9998	0.9987	0.9959	0.9938	0.9922	1.9529
0.6	0.4857	0.9028	0.989	0.9999	0.9988	0.9965	0.9946	0.9933	1.9438
0.7	0.5428	0.9217	0.9915	0.9999	0.999	0.9972	0.9957	0.9946	1.9358
0.8	0.6276	0.9439	0.9942	0.9999	0.9993	0.9979	0.9969	0.9962	1.9287
0.9	0.7619	0.9699	0.997	0.9999	0.9996	0.9989	0.9983	0.9979	1.9222

Table 1 continued

v = 4, n=24, 2n=N= 48									
ρ	α								
	1	1.3	1.6	1.9	2.2	2.5	2.8	3.1	α^*
-0.9	0.7101	0.9555	0.9922	0.9984	0.9997	0.9999	0.9999	0.9998	2.7796
-0.8	0.5639	0.9191	0.9856	0.9971	0.9996	0.9999	0.9999	0.9996	2.6911
-0.7	0.4773	0.8895	0.98	0.9962	0.9995	0.9999	0.9998	0.9994	2.616
-0.6	0.4213	0.8655	0.9754	0.9955	0.9995	0.9999	0.9997	0.9992	2.5547
-0.5	0.3833	0.8463	0.9717	0.9949	0.9995	0.9999	0.9996	0.9989	2.5057
-0.4	0.3569	0.8314	0.9688	0.9946	0.9995	0.9999	0.9995	0.9988	2.4668
-0.3	0.3389	0.8203	0.9667	0.9944	0.9995	0.9999	0.9994	0.9987	2.4358
-0.2	0.3271	0.8127	0.9653	0.9943	0.9995	0.9999	0.9993	0.9986	2.4109
-0.1	0.3205	0.8084	0.9647	0.9943	0.9996	0.9999	0.9992	0.9985	2.3906
0	0.3184	0.8072	0.9647	0.9944	0.9996	0.9999	0.9992	0.9984	2.3738
0.1	0.3207	0.8091	0.9653	0.9947	0.9997	0.9999	0.9992	0.9984	2.3598
0.2	0.3275	0.8142	0.9666	0.9949	0.9997	0.9999	0.9992	0.9985	2.3479
0.3	0.3395	0.8224	0.9685	0.9953	0.9997	0.9999	0.9992	0.9985	2.3378
0.4	0.3577	0.8341	0.9711	0.9958	0.9998	0.9999	0.9992	0.9986	2.329
0.5	0.3842	0.8495	0.9743	0.9963	0.9999	0.9999	0.9993	0.9987	2.3214
0.6	0.4225	0.8689	0.9781	0.9969	0.9999	0.9999	0.9994	0.9989	2.3147
0.7	0.4787	0.8928	0.9826	0.9976	0.9999	0.9999	0.9995	0.9991	2.3087
0.8	0.5655	0.922	0.9877	0.9983	0.9999	0.9999	0.9996	0.9994	2.3034
0.9	0.7115	0.9573	0.9935	0.9991	0.9999	0.9999	0.9998	0.9997	2.2987

Table 2: Values of WSRRs $R_{D(0.95)}(\rho)$ for slope rotatability for second order response surface designs with tri-diagonal correlation error structure using CCD for $\rho(-0.9 \leq \rho \leq 0.9)$ and fo $2 \leq v \leq 8$

v	α							
	1	1.3	1.6	1.9	2.2	2.5	2.8	3.1
2	-	0.7-0.9	0.7-0.9	-1.8	-1.8	-1.8	-1.8	-1.8
3	-	-	-1.8	-1.8	-1.8	-1.8	-1.8	-1.8
4	-	-0.9, 0.9	-1.8	-1.8	-1.8	-1.8	-1.8	-1.8
5	-	-	-1.8	-1.8	-1.8	-1.8	-1.8	-1.8
6	-	-	-1.8	-1.8	-1.8	-1.8	-1.8	-1.8
7	-	-	-1.8	-1.8	-1.8	-1.8	-1.8	-1.8
8	-	-	-1.8	-1.8	-1.8	-1.8	-1.8	-1.8

Note 1: Here indicates that the values of slope rotatability for second order response surface designs with tri-diagonal correlation error structure using CCD. For each value of α^*, the $M_V(D)$ is equal to 1.

Note 2: Measure of slope rotatability for second order response surface designs ($M_V(D)$) with tri-diagonal correlation error structure using CCD $\rho(-0.9 \leq \rho \leq 0.9)$ and for $2 \leq v \leq 4$ are available at the authors.

5 Conclusion

In this paper, the measure of slope rotatability for second order response surface designs with tri-diagonal correlationerror structure using CCD is studied. The degree of slope rotatability of the given design can be calculated for different values of $\rho(-0.9 \leq \rho \leq 0.9)$ and for $2 \leq v \leq 8$ (v number of factors). By increasing α and ρ values for different factors (v) the measure of slope rotatability values for second order response surface design withtri-diagonal correlation error structure using CCD are increased.

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